Conditional Probability

Problem: A math teacher gave two tests. 25% of the class passed both tests and 42% passed the first test. What is the probability that ~~[illegible]~~ that passed the first test also passed the second test?

Solution: So we want to know the probability that a student passed the first test and then went on to pass the second test. So student passed the ~~first~~ Second test GIVEN the student passed the first one. So this can be solved using conditional probability

Let A = event of passing second test

B = event of passing first test

By law of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

~~since A and B are independent events, (or we can assume) we can also apply the multiplicative rule to the $P(A \cap B)$ so~~

$$\cancel{P(A|B) = \frac{P(A).P(B)}{}}$$

$$P(A|B) = \frac{0.25}{0.42} = 0.6 \text{ or } 60\%$$

Problem : Conditional probability is applied in population studies, and to analyze census data. In one country 89.835 % of the female population live to age 60, while 57.062 % live up to 80 years. What is the probability that a woman who is 60 will live to age 80 ?

Solution : ~~Pf~~ we are trying to find

$$P(\text{live to } 80 \mid \text{lived to } 60)$$

applying conditional probability law,

$$P(\text{live to } 80 \mid \text{lived to } 60) = \frac{P(\text{live to } 80 \cap \text{lived to } 60)}{P(\text{lived to } 60)}$$

$$= \frac{0.57062}{0.89835}$$

$$= 0.6352 \text{ or } 63.52\%$$

Tennis - Venus Williams (data from ESPN.com)

Problem: A tennis player is allowed two serves on each point. Career stats show that Ms Williams gets her first serve in 75% of the time, resulting in her winning the point about 85% of the time. When she misses the first serve, her second serve is succesful 90% of the time, resulting in her winning the point 35% of the time.

a) what is the probability that she wins a point when serving?

b) If you know she won a point while serving, what is the probability that she made her first serve?

Soln: OK! lets gather the information using the correct nomenclature.

$P(\text{1st serve succesful}) = 0.75$

$P(\text{winnig point} \mid \text{1st serve succesful}) = 0.85$

$P(\text{2nd serve success.}) = 0.90$

$P(\text{winning point} \mid \text{2nd serve success}) = 0.35$

a) we want to know probability of winning a point when serving. But there are two scenarios here;

i/ winning point from succesful first serve

ii/ winning point from succesful second serve.

So we need to combine these. We invoke the law of total probability which is essentially a summation of all conditional probabilities for that event. So if event A is conditional on an B, and also in a different context condition on an event C., then

$$P(A) = P(B).P(A|B) + P(C).P(A|C) \quad \text{and so on}$$

So

a) $$P\left(\substack{\text{wins point when} \\ \text{Serving}}\right) = P\left(\substack{\text{1st Serve} \\ \text{Succes}}\right) . P\left(\substack{\text{winning} \\ \text{Point}} \middle| \substack{\text{1st Serve} \\ \text{Success}}\right)$$

$$+ P\left(\substack{\text{2nd Serve} \\ \text{Success}}\right) . P\left(\substack{\text{winning} \\ \text{point}} \middle| \substack{\text{2nd Serve} \\ \text{Success}}\right)$$

$$= (0.75)(0.85) + (0.90)(0.35)$$

$$= 0.9525$$

b) $$P\left(\substack{\text{1st Serve} \\ \text{Success}} \middle| \substack{\text{winning} \\ \text{Point} \\ \text{when Serve}}\right) = \frac{P(\text{1st Serve} \cap \text{win point when Serve})}{P(\text{win point when Serv})}$$

OK! now lest do some manipulation here!

$$P\left(\overbrace{\text{1st serve success}}^{A} \,\middle|\, \overbrace{\text{winning point when serving}}^{B}\right) = \frac{P\left(\overbrace{\text{1st serve success}}^{A} \cap \overbrace{\text{win point when serv}}^{B}\right)}{P(\underbrace{\text{win point when serving}}_{B})}$$

Or since in this ~~case~~ context we must be serving to actually win the ~~P(winning point~~ point, 'win point when serving' ≡ 'winning point'

So,

$$P\left(\overbrace{\text{1st serve succes}}^{A} \,\middle|\, \overbrace{\text{winning point}}^{B}\right) = \frac{P\left(\overbrace{\text{1st serve}}^{A} \cap \overbrace{\text{winpoint}}^{B}\right)}{P(\underbrace{\text{win point}}_{B})}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{— (1)}$$

but also

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = P(B|A)P(A) \quad \text{— (2)}$$

Substituting (2) into (1),

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

So we have

$$P\left(\text{1st serve success} \,\middle|\, \text{winning point}\right) = \frac{P\left(\text{winning point} \,\middle|\, \text{1st serve success}\right) . P(\text{1st serve success})}{P(\text{winning point})}$$

$$P\left(\begin{array}{c|c}\text{1st serve} & \text{winning} \\ \text{success} & \text{point}\end{array}\right) = \frac{0.85 \cdot 0.75}{\text{answer from (a)}} = \frac{0.85 \cdot 0.75}{0.9525} = 0.6693$$

So here is a funny way to make sense of this. If you tuned in late to the game and as soon as you tuned in the crowd was applauding Venus Williams after she had scored (and you missed the play), there is a probability of 0.67 or 2 chances in 3 that she scored off of a 1st serve.

And Now Some Tips & Tricks

At equation (1): $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ it is very tempting to do the following:

$$P(A|B) = \frac{P(A) \cdot \cancel{P(B)}}{\cancel{P(B)}}$$

Though this is theoretically correct for independent events, in working problems you will realize that in doing so you get stuck at a dead end. That is why it is far better to do the $P(A \cap B) \Leftrightarrow P(B \cap A)$ manipulation that resulted in equation (2). For the overwhelming majority of conditional probability problems I have encountered, the manipulation is the way to go. Please familiarize yourself with it and apply it whenever applicable or relevant.