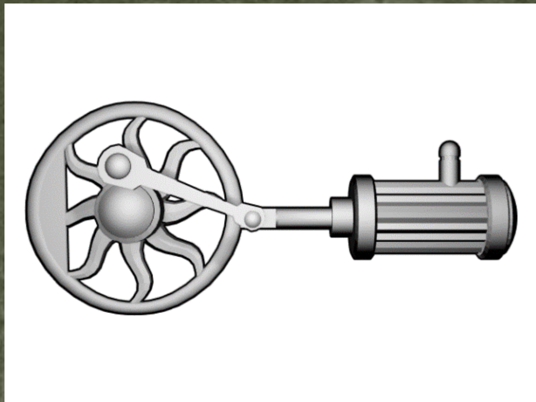
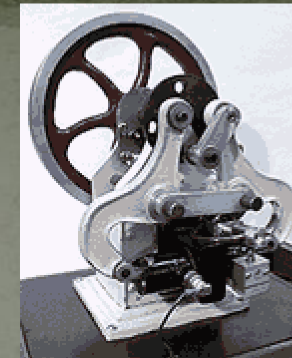
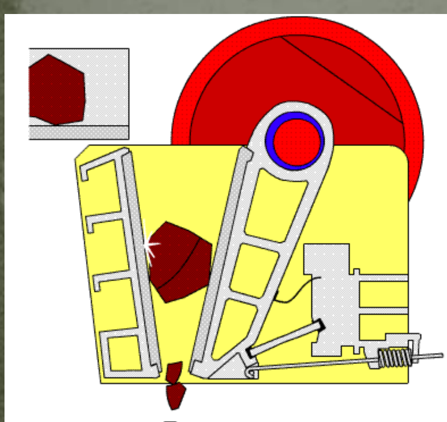
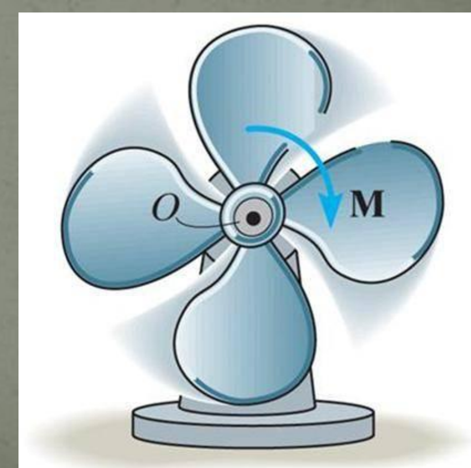




Moments of Inertia

Chapter 10

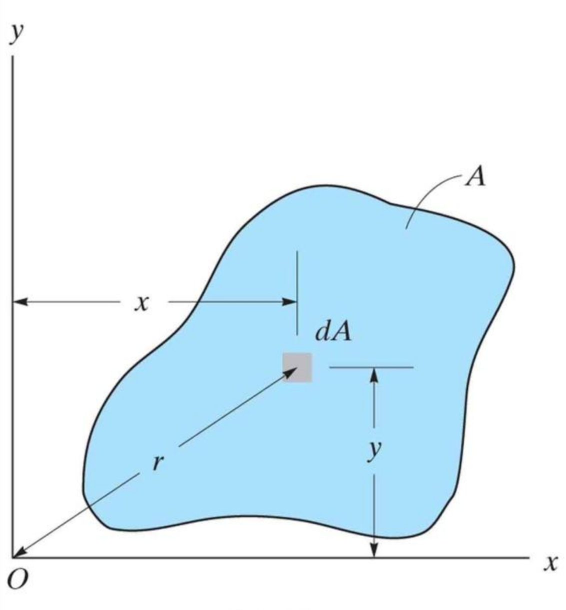




Overview

- Introduction
- Definition of Moment of Inertia
- Parallel Axis Theorem
- Radius of Gyration
- Composite Areas
- Product of Inertia
- Moments of Inertia for Inclined Axes
- Mass Moment of Inertia

Definition



- The *moment of inertia* of a differential area *dA* about an axis is given by

$$dI_x = y^2 dA$$

- For the entire area, the *moment of inertia* is obtained by integration

$$I_x = \int_A y^2 dA$$

- Likewise, the moment of inertia about the y-axis is

$$dI_y = x^2 dA \quad \Rightarrow \quad I_y = \int_A x^2 dA$$

Definition

- We may also take the moment of inertia about an axis perpendicular to the x-y plane through the point O.

$$dJ_O = r^2 dA$$

where *r* is the perpendicular distance from *dA* to the to the axis

- Integrating over the entire area we obtain the *polar moment of inertia* as

$$J_O = \int_A r^2 dA = I_x + I_y$$

- So the moment of inertia is always a positive value with unit m^4, mm^4, ft^4, in^4 etc etc

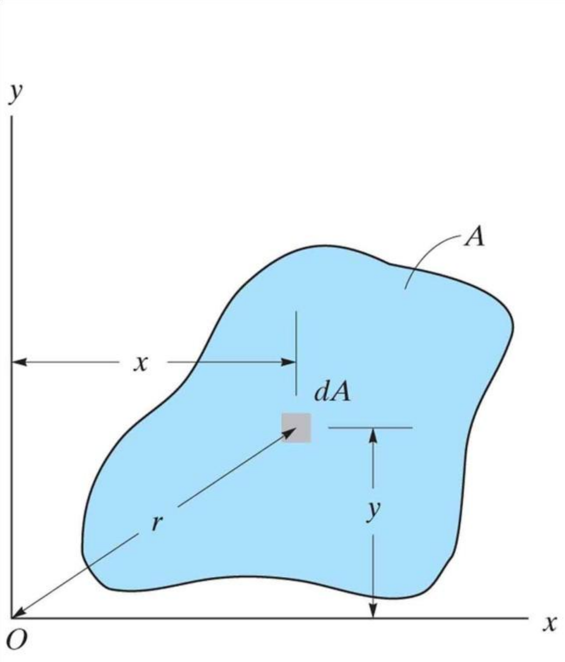


fig10_02.jpg

Tables and Manuals

- In practice we generally don't need to do the integration
- Most Engineering manuals present Moments of Inertia of common shapes that have been compiled in some chart or table
- The back cover of your textbook provides this information
- The full integration may only be necessary for "irregular" shapes

Parallel Axis Theorem

- The Parallel Axis Theorem enables us to find the moment of inertia about any axis parallel to an axis about which the moment of inertia is known
- Consider an axis passing through the centroid of the area

$$I_x = \int_A \left(y' + d_y\right)^2 dA$$

$$I_x = \int_A y'^2\, dA + 2d_y \int_A y' dA + d_y^2 \int_A dA$$

$$I_x = \underbrace{\int_A y'^2\, dA}_{centroidal\ \mathrm{MOI}} + 2d_y \underbrace{\int_A y' dA}_{\mathrm{zero}} + d_y^2 \int_A dA$$

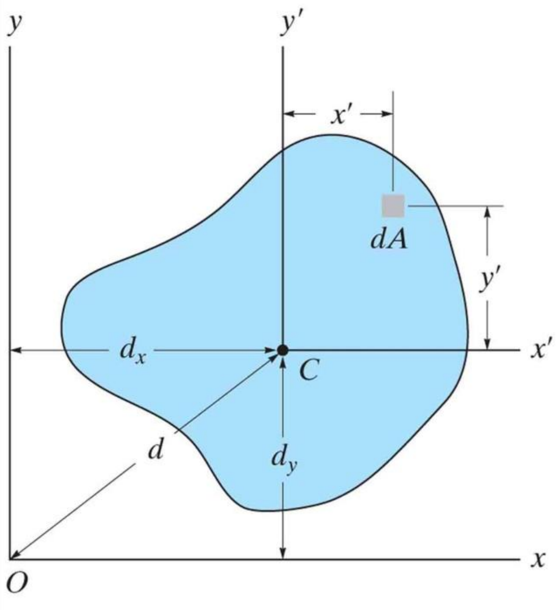


fig10_03.jpg

Parallel Axis Theorem

- Therefore

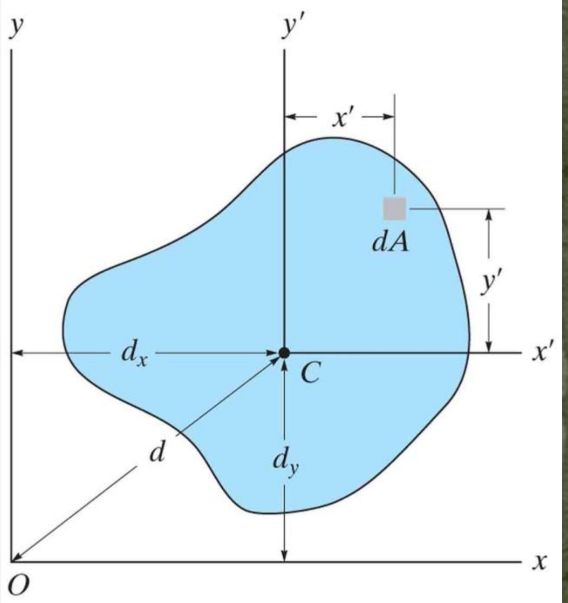
$$I_x = \bar{I}_{x'} + Ad_y^2$$

- For the y-axis

$$I_y = \bar{I}_{y'} + Ad_x^2$$

- For the polar moment of inertia

$$J_O = \bar{J}_C + Ad^2$$



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Radius of Gyration

- If the moments and areas are **known**, the *radius of gyration* about an axis is given by

$$k_x = \sqrt{\frac{I_x}{A}}$$

- For the y-axis

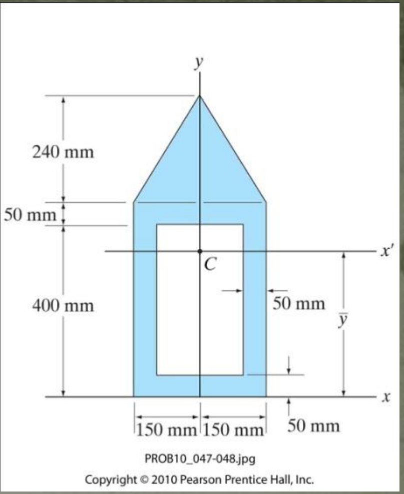
$$k_y = \sqrt{\frac{I_y}{A}}$$

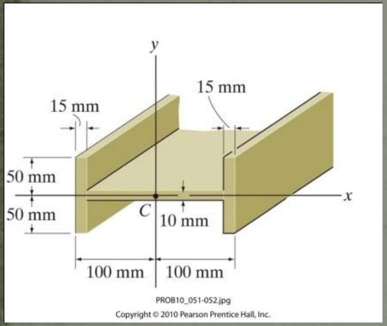
- For the polar moment of inertia

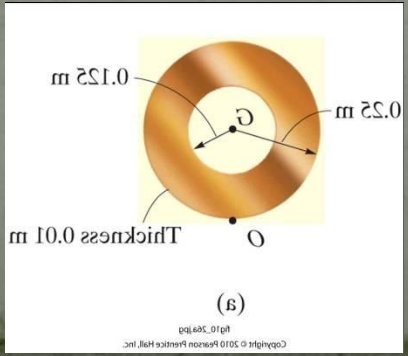
$$k_O = \sqrt{\frac{J_O}{A}}$$

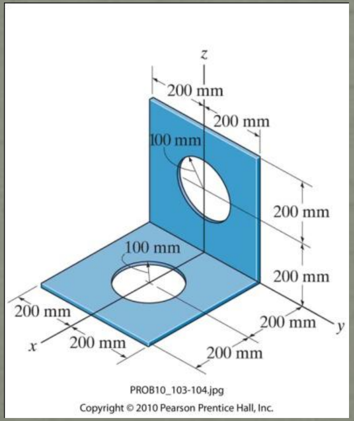
Composite Areas

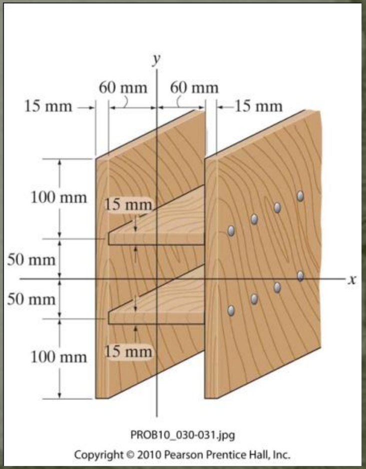
- A *composite area* is a series of connected simpler shapes
- The moment of inertia of a composite area is the algebraic sum of the moments of inertia of the constituent parts







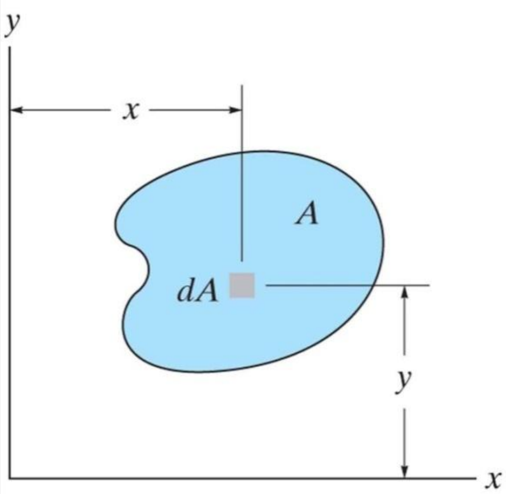




Product of Inertia

- The *product of inertia* with respect to the x-axis (in other words an infinitesimally small dimension of *dA* in the x-axis) is defined as

$$I_{xy} = \int_A xydA$$



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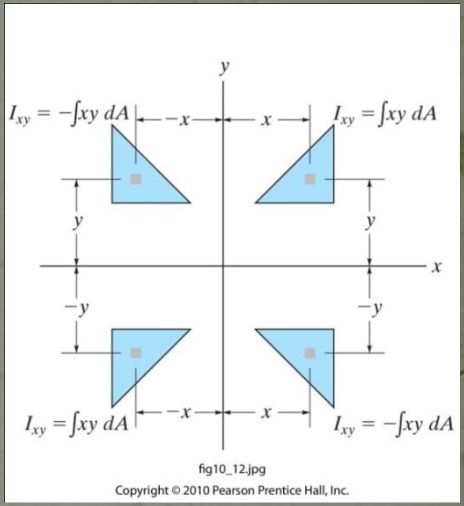
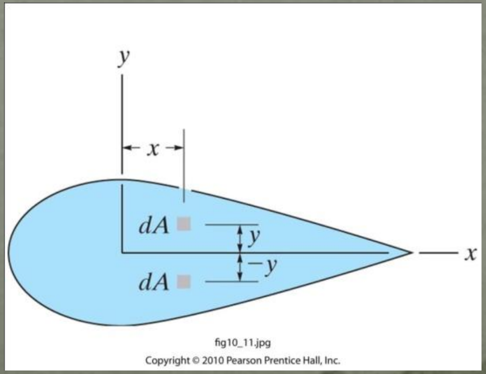
- Note that if *dA* is formed from infinitesimally small dimensions in x and in y, the a double integral will be involved
- *Product of inertia* enables us determine the maximum and minimum moments of inertia for an area

Product of Inertia - Properties

- Product inertia may yield negative values
- The sign of the product of inertia will depend on which quadrant the area is located in
-



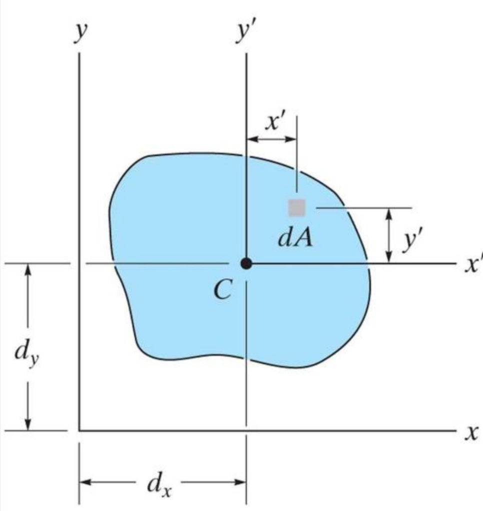
- Product inertia about an axis of symmetry will be zero



Product of Inertia

- Parallel Axis Theorem:
  It can be shown that (please see text for full proof)

$$I_{xy} = \bar{I}_{x'y'} + A d_x d_y$$



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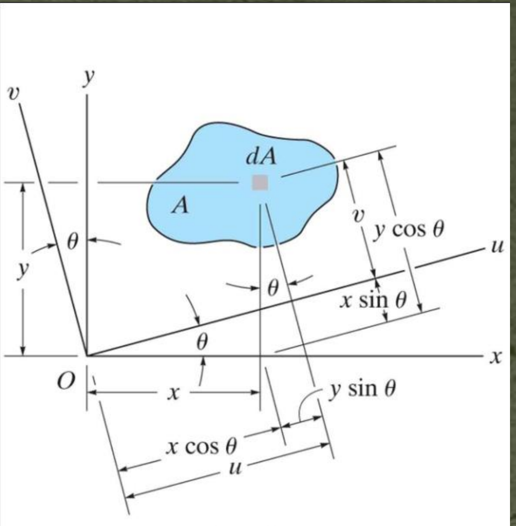
Inclined Axes



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- Let's say I_x, I_y, and I_{xy} are known.
- Now we rotate our x-y reference frame through an angle θ to obtain orthogonal axes u and v respectively
- It can be shown that (see text for full proof)

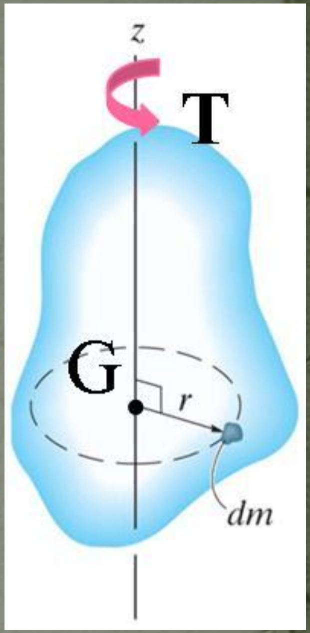
$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{uv} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$J_O = I_u + I_v = I_x + I_y$$

Mass Moment of Inertia



- Consider a rigid body with a center of mass at G free to rotate about the z axis, which passes through G.
- if we apply a torque T about the z axis to the body, the body begins to rotate with an angular acceleration α.
- It turns out that T and α are related by the equation

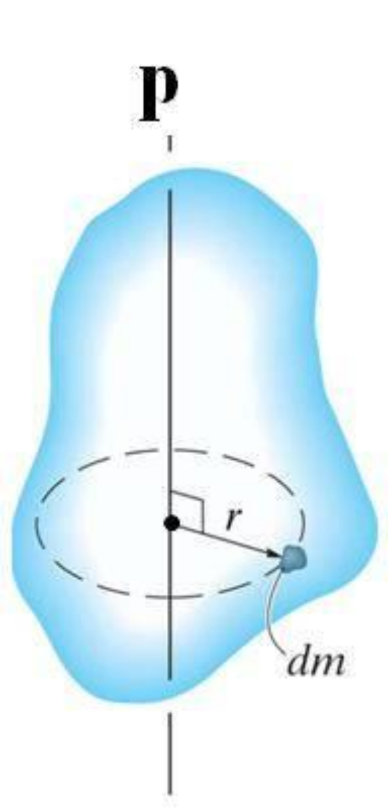
  $T = I\,\alpha$ (analogous to $F = ma$)
- I is the *mass moment of inertia (MMI)* about the z axis and it is a measure of the resistance to the body to the angular motion

Definition of MMI

- The MMI about the P axis is defined as

  $I = \int_m r^2 \, dm,$

  where *r*, is the "moment arm," aka the perpendicular distance from the axis to the arbitrary element dm.
- The MMI is always a positive quantity and has a unit of $kg \cdot m^2$ or $slug \cdot ft^2$.



Tables and Manuals

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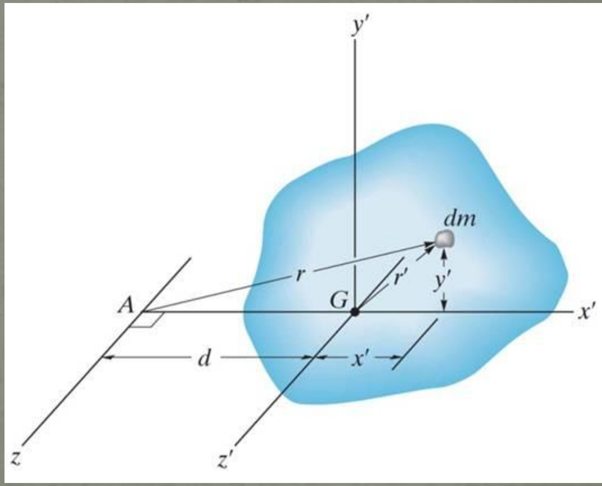
- Most Engineering manuals present Moments of Inertia of common shapes that have been compiled in some chart or table
- The back cover of your textbook provides this information
- Full integration may be necessary for irregularly shaped bodies

Parallel Axis Theorem

- The MMI is defined about a specific axis.
- To find the MMI of the object about an axis parallel to an axis for which the MMI is already known we can use the Parallel Axis Theorem, which states

$$I_z = I_G + (m)(d)^2$$



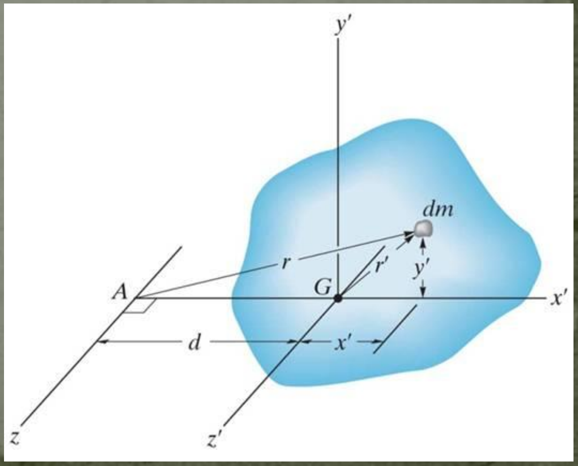
Radius of Gyration

- An alternate definition of MMI about an axis is

  $$k = \sqrt{(I / m)}$$

  where k is called the *radius of gyration*

- Finally, the MMI can be obtained by integration or by the method for *composite bodies*. The latter method is easier for many practical shapes.



Questions & Comments