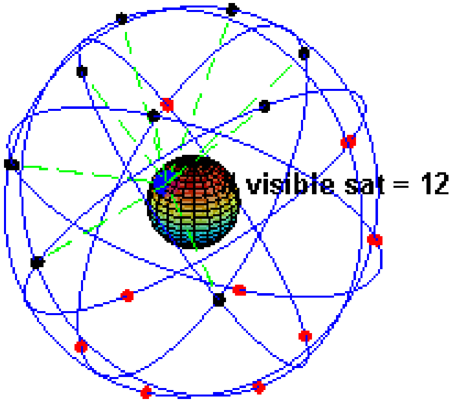
Kinematics

Curvilinear Motion



Christiaan Hugens

Overview

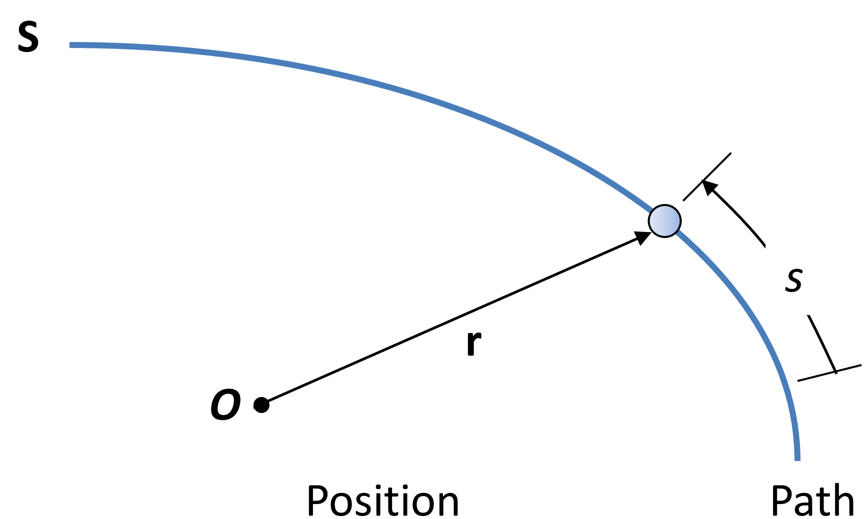
- General Curvilinear Motion
- Curvilinear Motion: Rectangular Components
- Projectile Motion

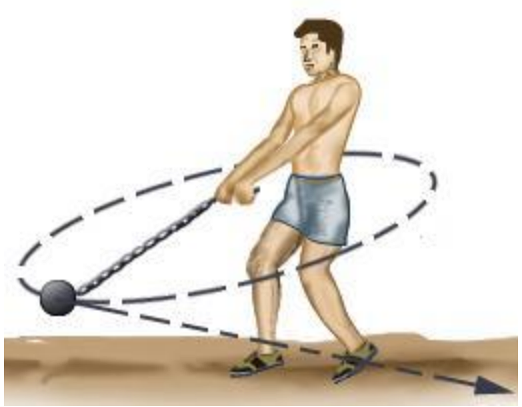
General Curvilinear Motion

- Position

We can consider an object's position on a circular path as a *position vector*

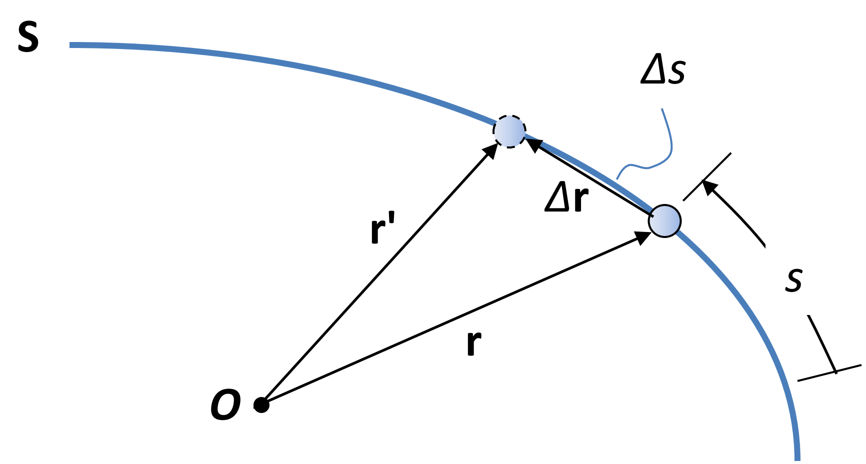
$$\mathbf{r} = \mathbf{r}(t)$$





Curvilinear Motion - Displacement

- Displacement:
  the change in the particle's position.
- Lets say the particle moves Δs along path in time interval Δt
- New position, $\mathbf{r}' = \mathbf{r} + \Delta\mathbf{r}$
- *Displacement*, $\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$



Displacement

Curvilinear Motion - Velocity

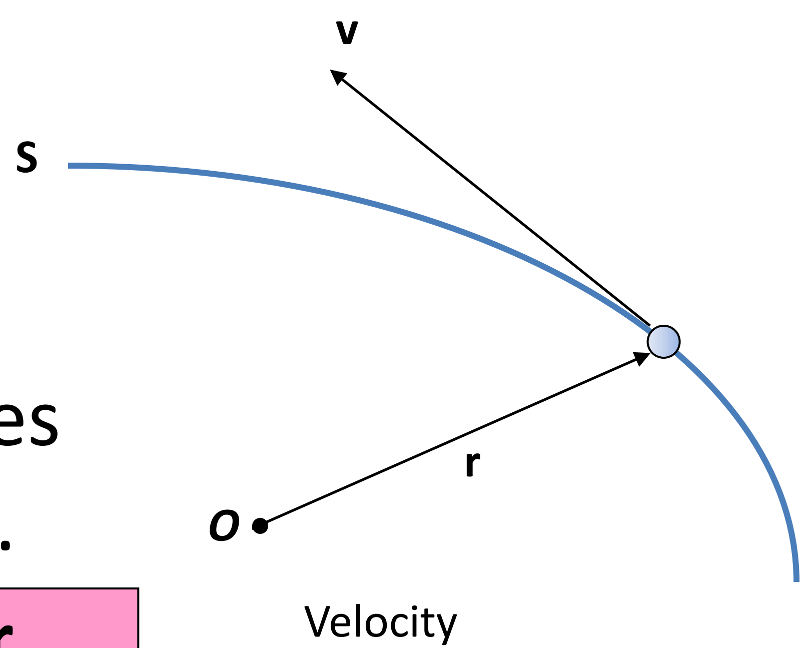
- Velocity
  During interval Δt,
  the *average velocity*

$$\mathbf{v}_{avg} = \frac{\Delta \mathbf{r}}{\Delta t}$$

As Δt $\rightarrow$ 0, Δ**r** approaches the tangent to the curve.

$$\mathbf{v} = \lim_{\Delta t \to 0}\left(\frac{\Delta \mathbf{r}}{\Delta t}\right)$$ or $$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

v is the *instantaneous velocity*



Curvilinear Motion - Speed

- Note that **v** is tangential to the curve.
- The magnitude of **v** is called the *speed*
- As $\Delta t \rightarrow 0$, $\Delta \mathbf{r} \rightarrow \Delta s$. So the speed

$$v = \lim_{\Delta t \to 0}\left(\frac{\Delta \mathbf{r}}{\Delta t}\right) = \lim_{\Delta t \to 0}\left(\frac{\Delta s}{\Delta t}\right)$$

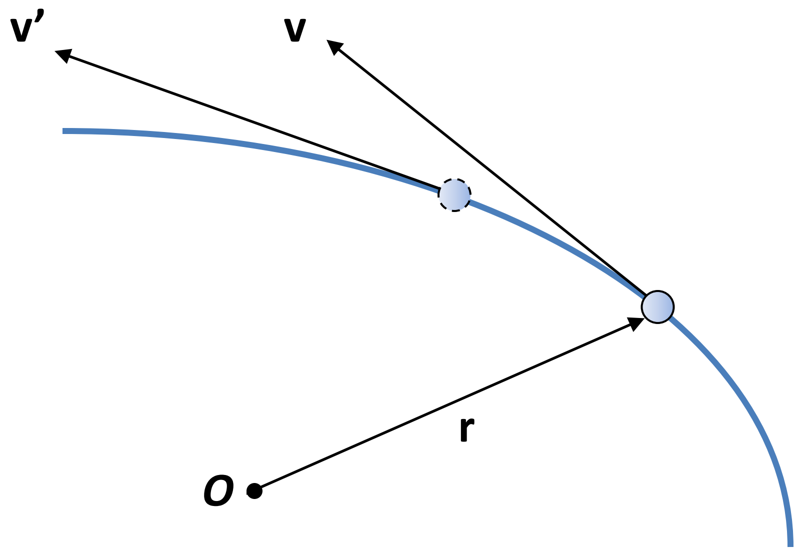
$$v = \frac{ds}{dt}$$

Curvilinear Motion -Acceleration

- If velocity is **v** at time t, and **v'** at time t+Δt, then *average acceleration*

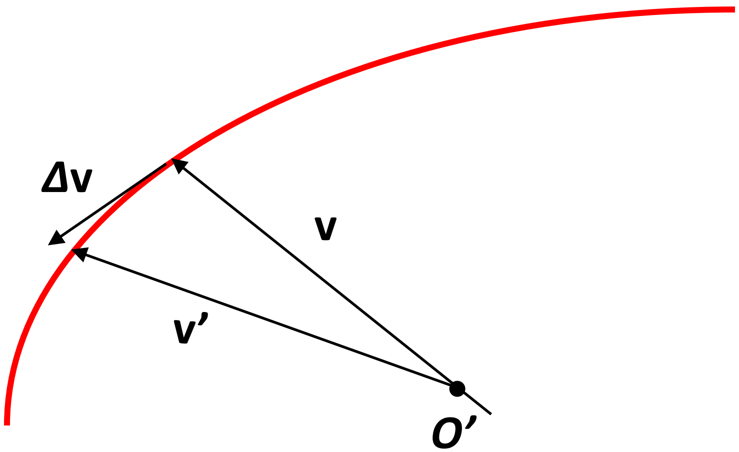
$$\mathbf{a}_{avg} = \frac{\Delta \mathbf{v}}{\Delta t}$$

where Δ**v** = **v'** - **v**



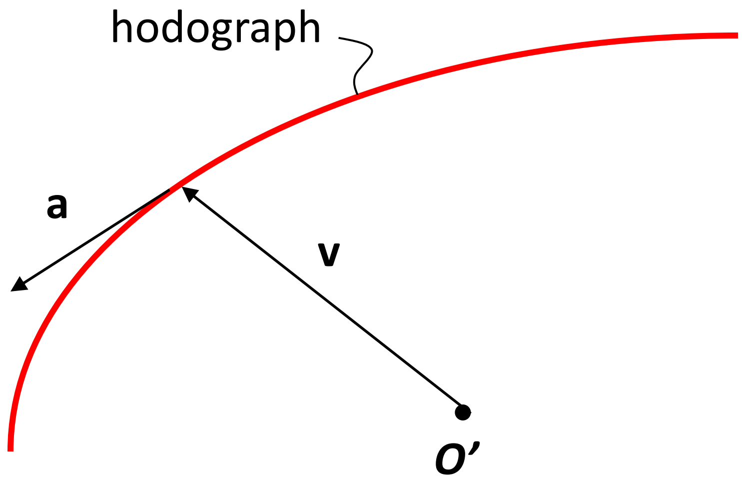
Velocity

Curvilinear Motion-Acceleration



- If we plot vectors **v** and **v’** to scale from a common origin, curved path touching their arrowheads is called *hodograph*.
- The hodograph is analogous to the *path s* for the position vectors.

Curvilinear Motion - Acceleration



- If Δt approaches 0, then Δv will approach the tangent to the hodograph.
- *Instantaneous acceleration*

$$\mathbf{a} = \lim_{\Delta t \to 0} \left(\frac{\Delta \mathbf{v}}{\Delta t} \right) \quad \text{or}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

Curvilinear Motion - Acceleration

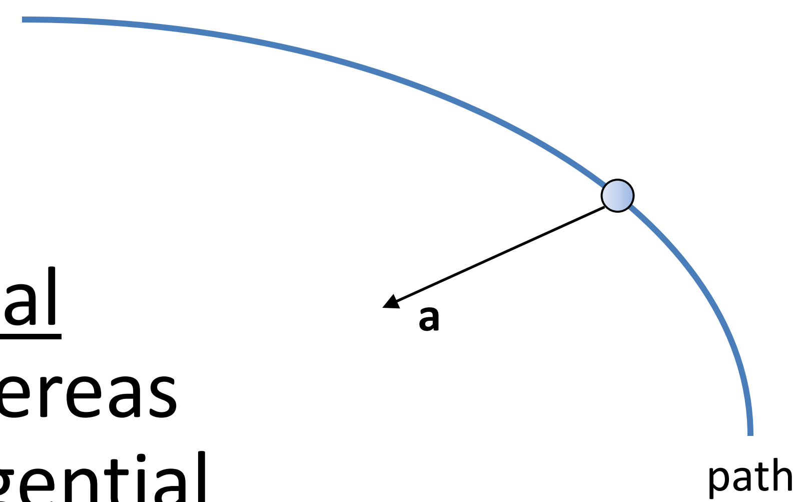
- Substituting the instantaneous velocity into the instantaneous acceleration equation,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$$

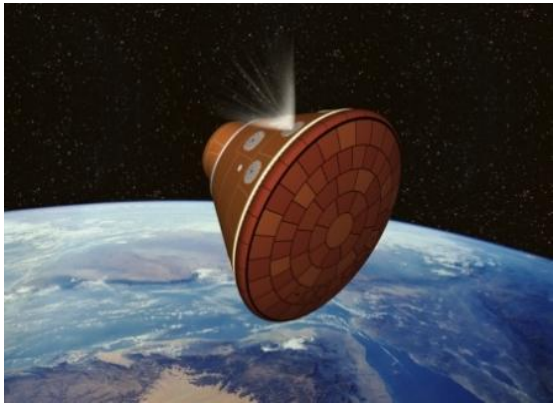
Curvilinear Motion-Acceleration

- It is pertinent to note that acceleration acts tangential to the hodograph, but generally not tangential to the path of motion *s*
- Velocity is always tangential to the path of motion, whereas acceleration is always tangential to the hodograph



Conclusion

- Examples



Jean-Baptiste le Rond d'Alembert

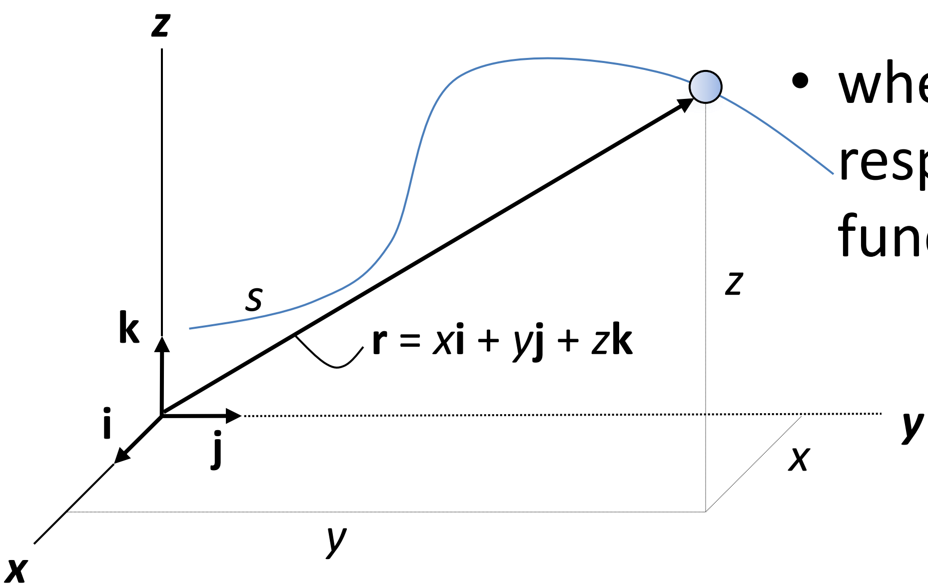
KINEMATICS

Curvilinear Motion: Rectangular Components

Curvilinear Motion: Rectangular Components - Position

- Consider the particle is moving in this 3-d frame of reference. The *position vector*

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



Position

- where **i**, **j**, **k** are unit vectors in the respective directions, and *x*, *y*, *z* are functions of time

Rectangular Components - Position

- At any instant, the *magnitude* of **r**

$$|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

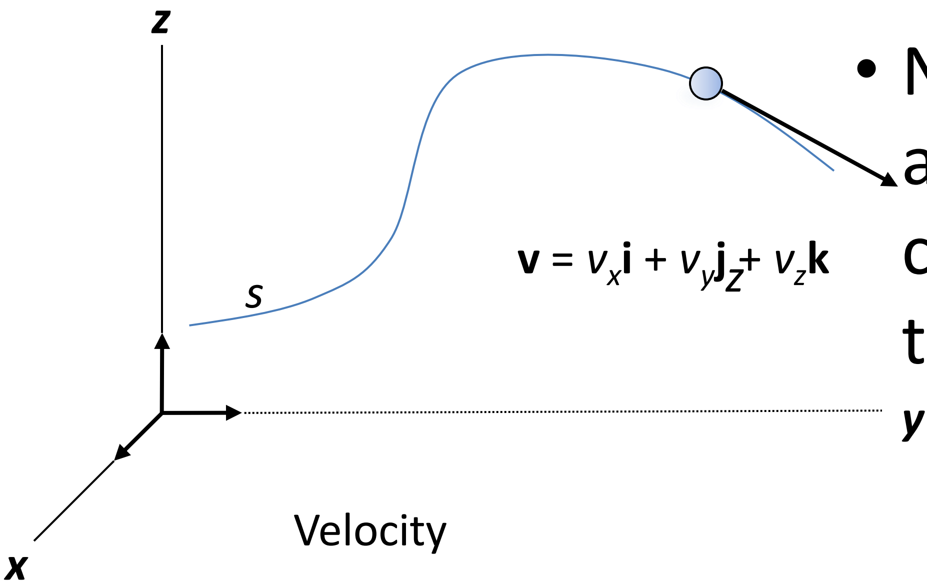
- And the *direction* of the vector **r** is specified by the unit vector

$$\mathbf{u}_r = \frac{1}{|\mathbf{r}|}.\mathbf{r}$$

Rectangular Components - Velocity

- The derivative of **r** with respect to time yields the velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(x\mathbf{i}\right) + \frac{d}{dt}\left(y\mathbf{j}\right) + \frac{d}{dt}\left(z\mathbf{k}\right)$$



- Note that the magnitude and direction of each component is a function of time.

Rectangular Components - Velocity

- So for the **i** component, for example, we must apply the product rule of differentiation

$$\frac{d}{dt}(x\mathbf{i}) = x\frac{d\mathbf{i}}{dt} + \frac{dx}{dt}\mathbf{i}$$

- The first term will be zero if we keep our frame of reference fixed so that **i** does not change with time

$$\frac{d}{dt}(x\mathbf{i}) = \frac{dx}{dt}\mathbf{i} = \dot{x}\mathbf{i} \text{ or } v_x\mathbf{i}$$

Rectangular Components - Velocity

- So for all three components,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

or

$$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$

- The *magnitude* of the velocity is

$$|\mathbf{v}| = v = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

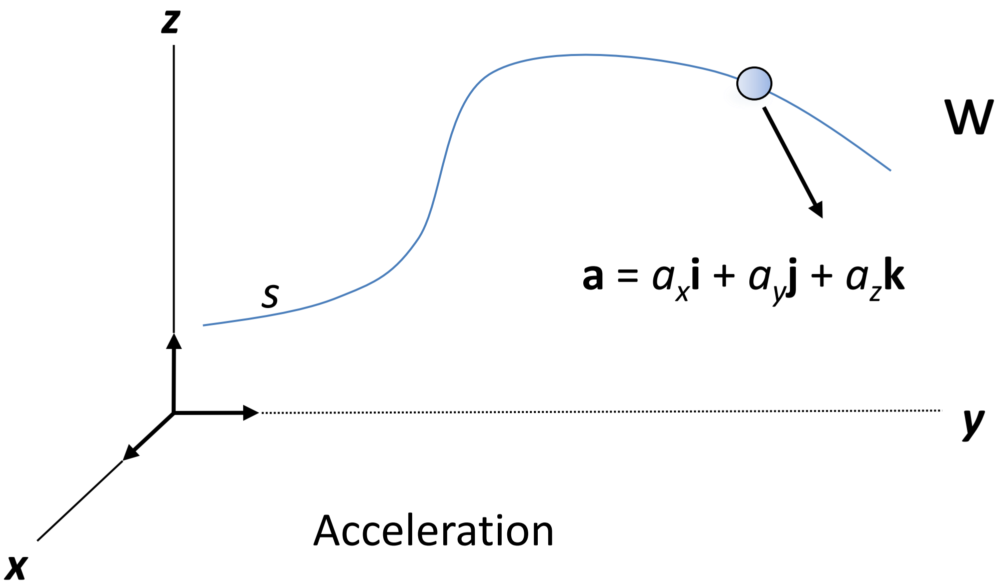
- And the *direction* of the velocity is

$$\mathbf{u}_{\mathrm{v}} = \frac{1}{|\mathbf{v}|}.\mathbf{v}$$

Rectangular Components - Acceleration

- Acceleration is the first derivative of velocity with respect to time. Or the second derivative of position with respect to time

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$



Acceleration

where

$$a_x = \dot{v}_x = \ddot{x}$$

$$a_y = \dot{v}_y = \ddot{y}$$

$$a_z = \dot{v}_z = \ddot{z}$$

Rectangular Components - Acceleration

- The acceleration has *magnitude*

$$|\mathbf{a}| = a = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

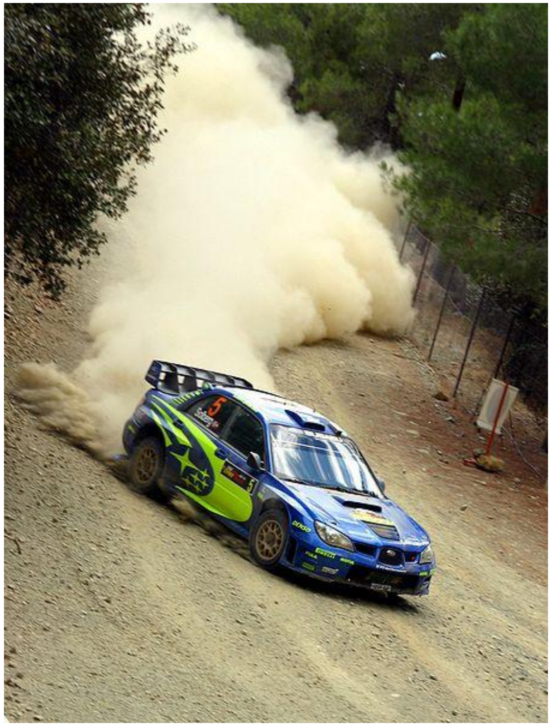
- The acceleration has *direction*

$$\mathbf{u}_a = \frac{1}{|\mathbf{a}|}.\mathbf{a}$$

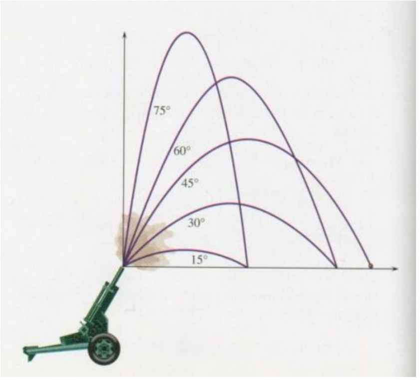
- Note that acceleration is not tangential to the path of motion whereas the velocity always is.

Questions & Comments

- How did that go ?



- Examples



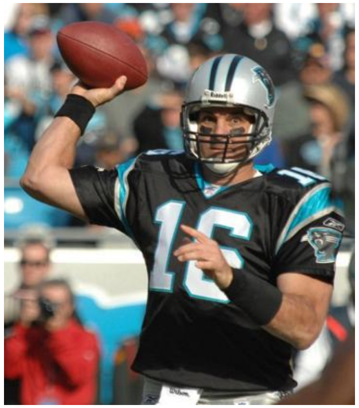
KINEMATICS

Projectile Motion

Niccolo Fontana Tartaglia

What is a Projectile?

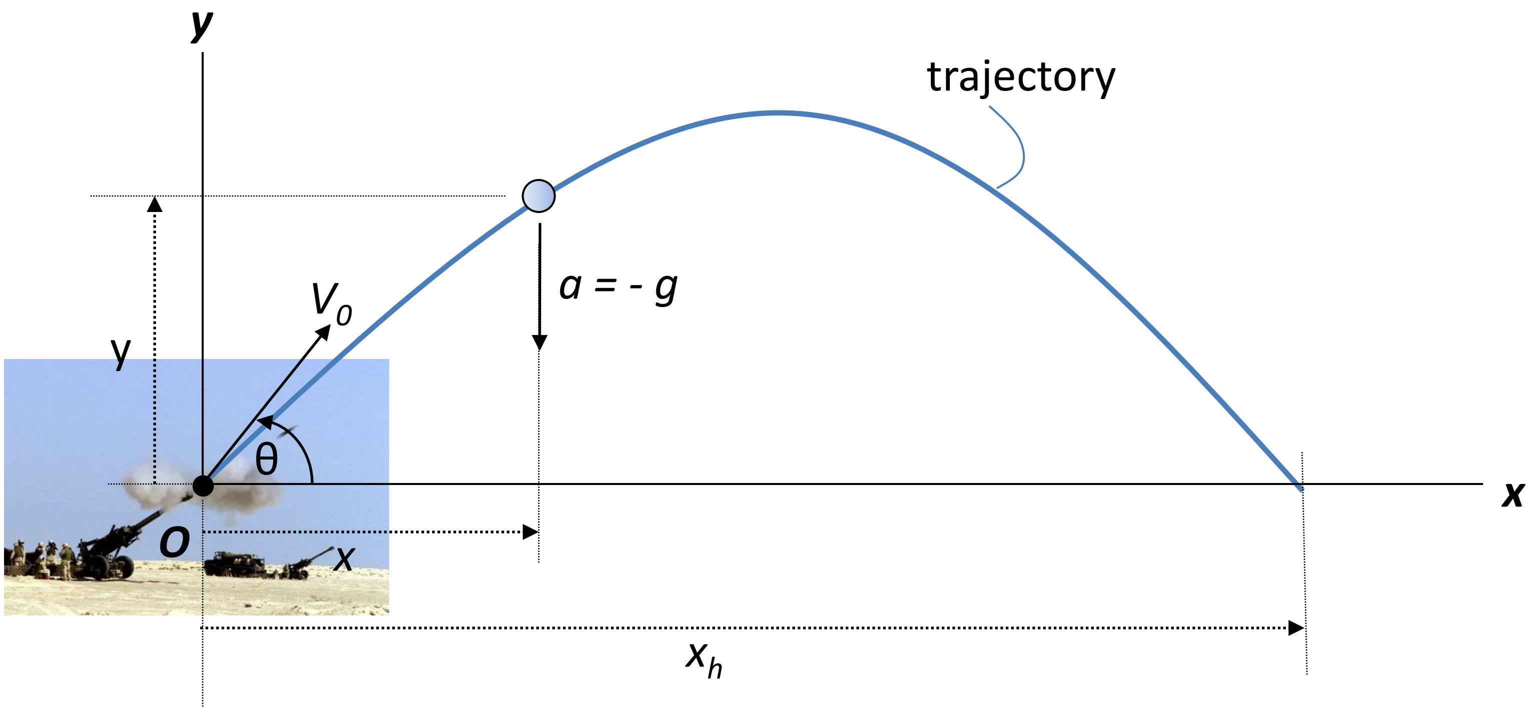
- an object projected into the air at an angle, and once projected continues in motion by its own inertia and is influenced only by the downward force of gravity
- Examples: football being kicked or thrown, an athlete long jumping, the motion of a cannon ball



Brief History

- Niccolo Tartaglia (1500 – 1557), realized that projectiles actually follow a curved path.
- Yet no one knew what that path was.
- Galileo (1564 – 1642) accurately described projectile motion by showing it could be analyzed by separately considering the horizontal and vertical components of motion.
- Galileo concluded that the path of *any* projectile is a parabola

Analysis



Acceleration

- We need six equation to completely describe projectile motion
- Horizontal component of motion (acceleration);

$$\boldsymbol{a_x = 0} \qquad \textbf{(1)}$$

- Vertical component of motion: the force of gravity will cause the body to fall towards the ground. (9.8 m/s^2 or 32.174 $ft/\ s^2$).

$$\boldsymbol{a_y = -g} \qquad \textbf{(2)}$$

Velocity

- $a_x = 0$.

This implies the horizontal component of velocity is constant from the time of projection to the time of impact with the ground

$$V_x = V_{xo} = V_o cos\theta \qquad (3)$$

Velocity

- Vertical component of velocity

  $a_y = -g$

- We have constant acceleration (i.e. continuous motion)
- From the 1st equation of motion:

$$v = v_0 + a_c\, t \quad \text{or} \quad V_y = V_0 + a_c\, t$$

$$V_y = V_{0y} + a_y t$$

$$V_y = V_{0y} - gt$$

$$V_y = V_0 \sin\theta - gt \qquad (4)$$

Displacement

- Horizontal component:

distance = velocity * time

$$x = V_x t$$

$$x = V_0 \cos\theta \; t \qquad (5)$$

Displacement

- Vertical component: Applying the 2nd equation of motion

$$s = ut + \tfrac{1}{2} at^2$$

$$s_y = V_{0y} t + \tfrac{1}{2} a_y t^2$$

$$y = V_0 \sin\theta \, t - \tfrac{1}{2} g t^2 \qquad (6)$$

- The exponent of the time term confirms the parabolic shape of the *trajectory*

Conclusion

- We have now fully described the projectile motion

- Examples