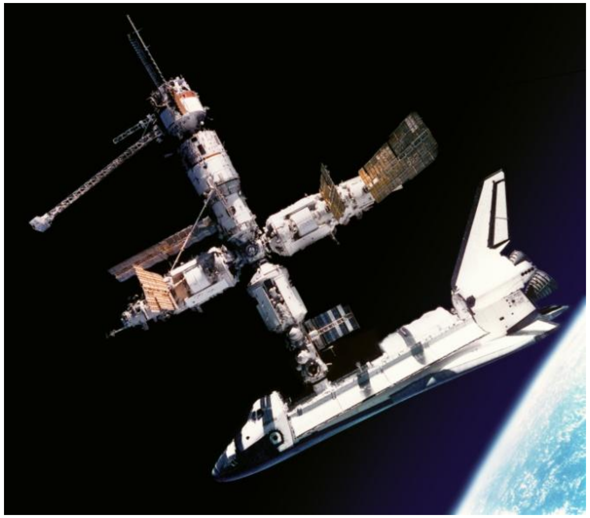


Johannes Kepler

Kinematics

Curvilinear Motion

Overview

- Curvilinear Motion: Normal and Tangential Components
- Curvilinear Motion: Cylindrical Coordinates

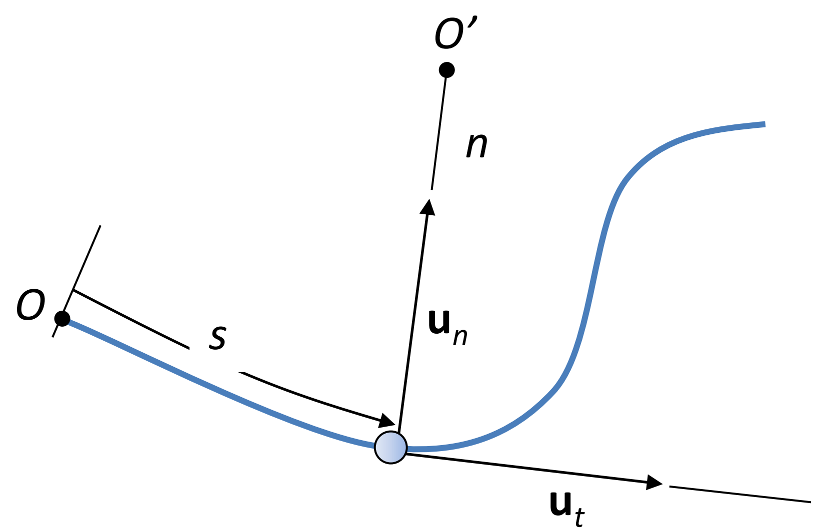
Curvilinear Motion: Normal and Tangential Components

Introduction

- For a particle moving along a **known curvilinear path**
- If we take the position at an instant as the origin
- We can define coordinate axes tangential, and normal to the path

Normal and Tangential Components

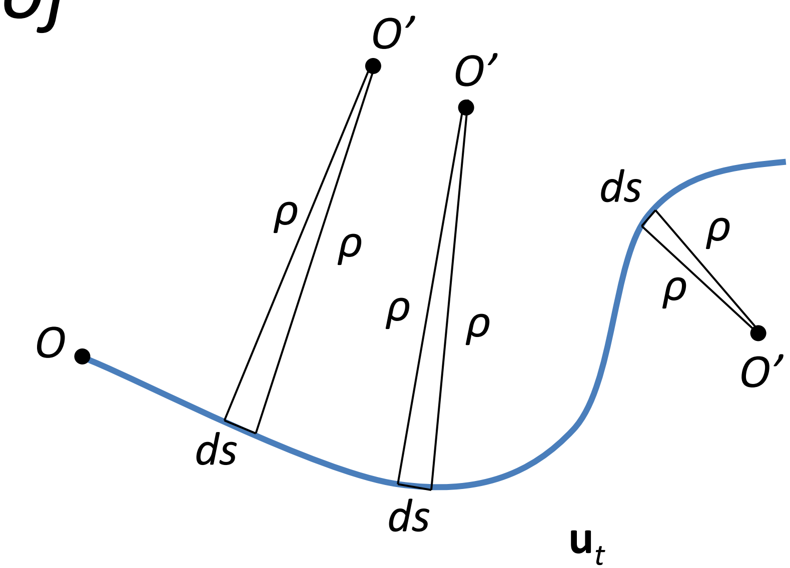
- Consider the particle at *s* from a fixed point *O*
- *t*-axis is tangent to the curve. *n*-axes is normal to the curve
- $\mathbf{u}_t$ and $\mathbf{u}_n$ are the respective unit vectors and are perpendicular to each other



Position

Normal and Tangential Components

- We may view the path as consisting of a series of infinitesimally small arcs of length *ds*, *radius of curvature ρ*, with center *O'*
- The *n*-axis is positive towards the *center of curvature*.
- The plane containing the *n* and *t* axes is called the *embracing* or *osculating plane*



Radius of curvature

Normal and Tangential Components

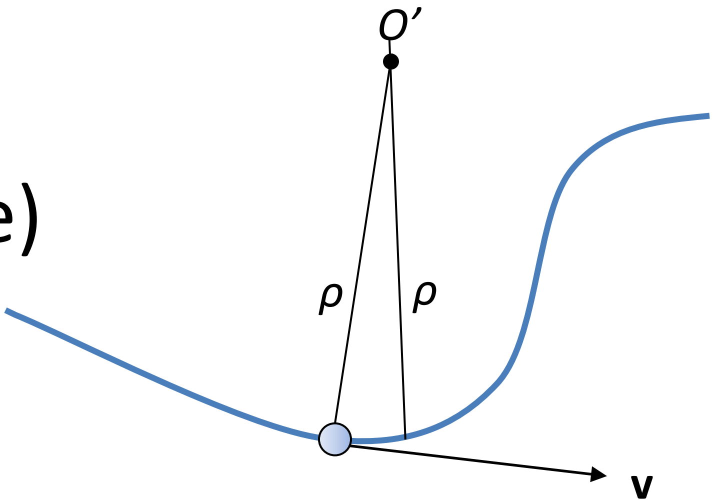
Velocity

- *Direction*: always *tangential to the path*
- Magnitude: Since $s = s(\text{time})$

$$v = \frac{ds}{dt} \quad \text{so} \quad \boxed{\mathbf{v} = v\mathbf{u}_t}$$

where

$$\boxed{v = \dot{s}}$$



Velocity

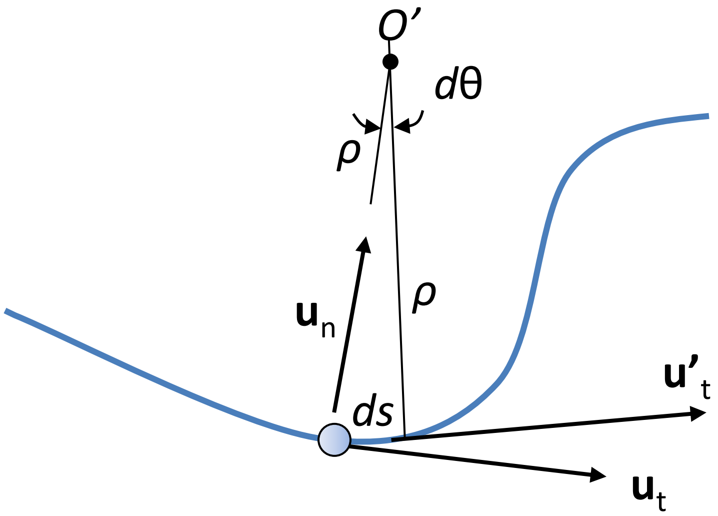
Normal and Tangential Components

Acceleration

- As can be seen velocity is continually changing as the particle traverses the curved path
- Acceleration is the rate of change velocity with respect to time

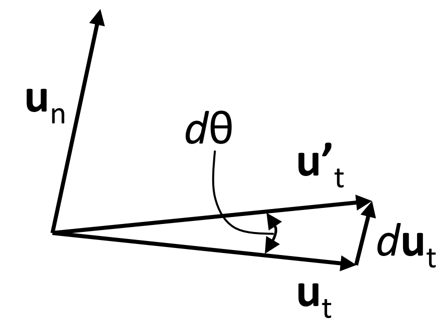
$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \frac{d(v\mathbf{u}_t)}{dt}$$

$$\boxed{\mathbf{a} = \dot{v}\mathbf{u}_t + v\dot{\mathbf{u}}_t}$$



Normal and Tangential Components

- Lets us redraw the velocity unit vectors at the infinitesimal scale



$$\mathbf{u}_t' = \mathbf{u}_t + d\mathbf{u}_t$$

  and for small angles $d\theta = \dfrac{d\mathbf{u}_t}{\mathbf{u}_t}$ or $d\mathbf{u}_t = d\theta \mathbf{u}_t$
  but note that $d\mathbf{u}_t$ is in the direction of a unit vector in the normal direction, so we can replace the unit vector, thus $d\mathbf{u}_t = d\theta \mathbf{u}_n$
- And the derivative with respect to time becomes

$$\dot{\mathbf{u}}_t = \dot{\theta} \mathbf{u}_n$$

Normal and Tangential Components

- From the properties of an arc we know that

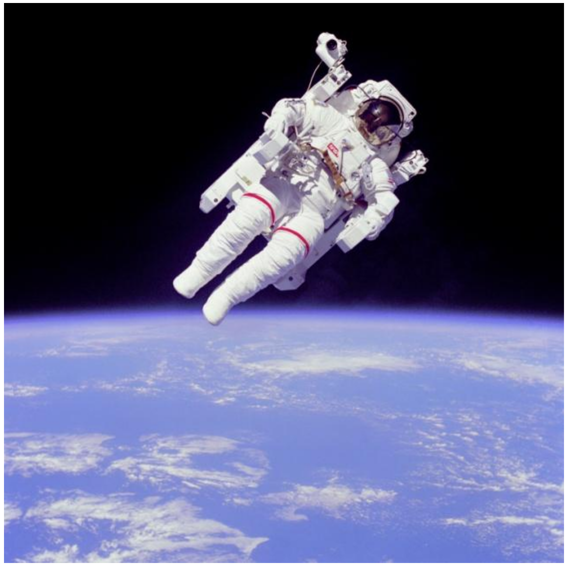
$$ds = \rho\, d\theta \qquad \text{(go back two slides)}$$

so likewise, with respect to time,

$$\dot{\theta} = \dot{s}/\rho$$

so

$$\dot{\mathbf{u}}_t = \dot{\theta}\,\mathbf{u}_n = \frac{\dot{s}}{\rho}\mathbf{u}_n = \frac{v}{\rho}\mathbf{u}_n$$



Normal and Tangential Components

- We can consider acceleration as having two components, one tangential and the other normal

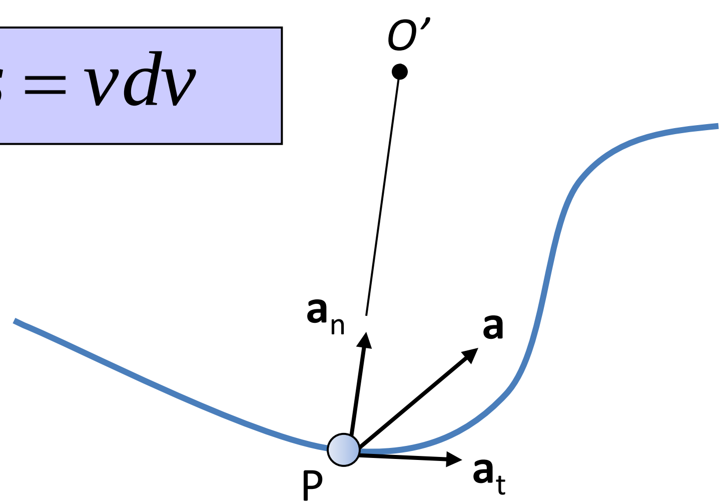
$$\mathbf{a} = a_t \mathbf{u}_t + a_n \mathbf{u}_n$$

- Comparing with original equation (Slide 7)

$$a_t = \dot{v}$$ or $$a_t ds = v dv$$

and

$$a_n = \frac{v^2}{\rho}$$

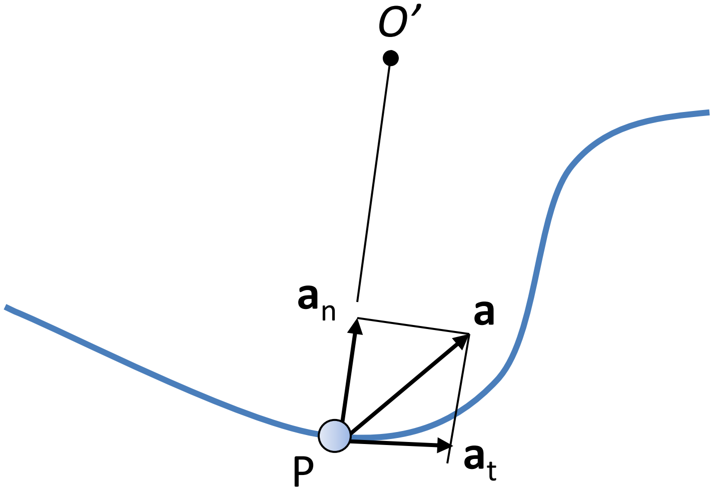


Acceleration

Normal and Tangential Components

- The magnitude of the acceleration is therefore

$$a = \sqrt{a_t^2 + a_n^2}$$



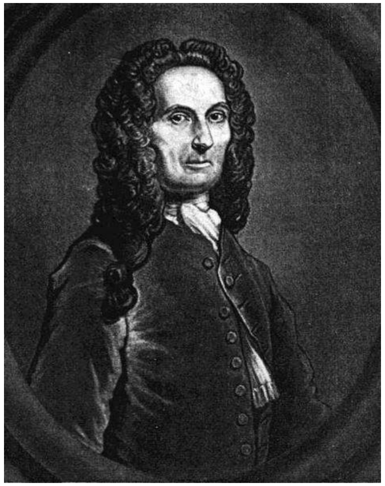
Acceleration

Conclusion

- Examples

Curvilinear Motion: Cylindrical Coordinates

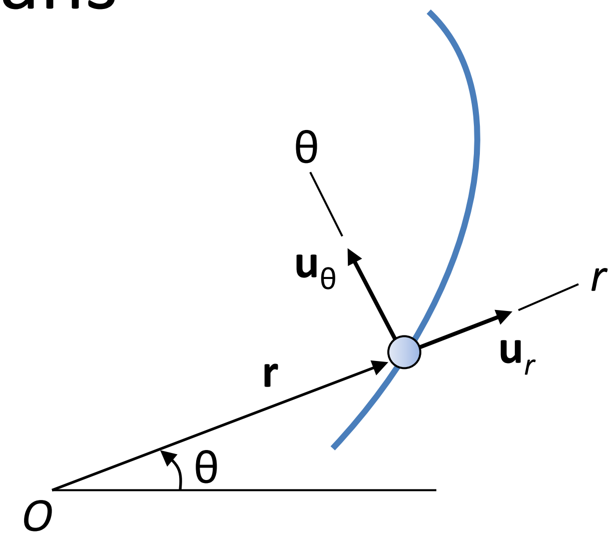
- In some cases the motion of a particle is constrained on a path amenable to analysis using cylindrical coordinates
- If the motion is restricted to a plane, then we can use polar coordinates



Abraham de Moivre

Polar Coordinates

- Consider a system where we locate a particle by a *radial coordinate* r, which extends from an origin *O*, and an angle θ (also called transverse coordinate) in radians measured counterclockwise form a fixed reference line to the axis of *r*.
- $\mathbf{u}_\theta$ and $\mathbf{u}_r$ are unit vectors in the directions of increasing θ and *r* respectively



Position

Position & Velocity

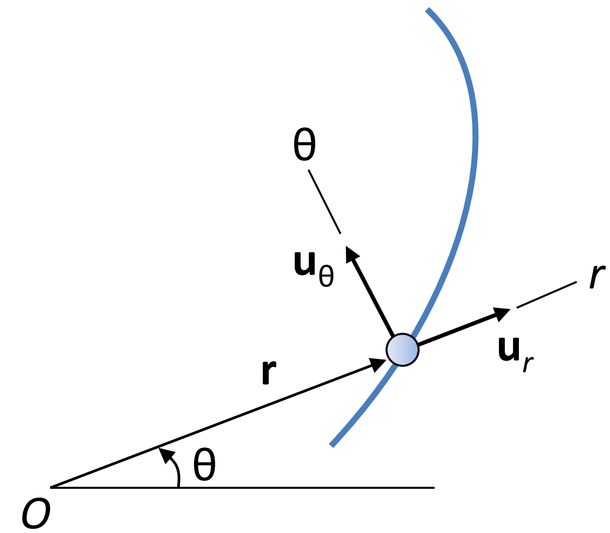
- At any instant, we can define the position vector of the particle as

$$\mathbf{r} = r\mathbf{u}_r$$

- The instantaneous velocity is the time derivative of **r**

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{u}_r + r\dot{\mathbf{u}}_r$$

- $\dot{\mathbf{u}}_r$?

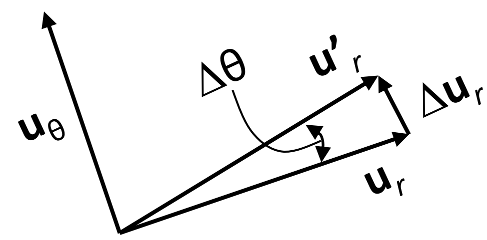


Position

Velocity

- A change in *r* over a time interval *Δt* will not result in a change of the unit vector *Δr*
- Rather a change *Δ*θ over an interval *Δt* will cause a change *Δr*

$$\mathbf{u}_r' = \mathbf{u}_r + \Delta \mathbf{u}_r$$



- For small angles $\Delta \mathbf{u}_r \approx \Delta \theta$
  and

$$\Delta \mathbf{u}_r = \Delta \theta \, \mathbf{u}_\theta$$

Velocity

$$\dot{\mathbf{u}}_r = \lim_{\Delta t \to 0}\left(\frac{\Delta \mathbf{u}_r}{\Delta t}\right) = \left(\lim_{\Delta t \to 0}\frac{\Delta \theta}{\Delta t}\right)\mathbf{u}_\theta$$

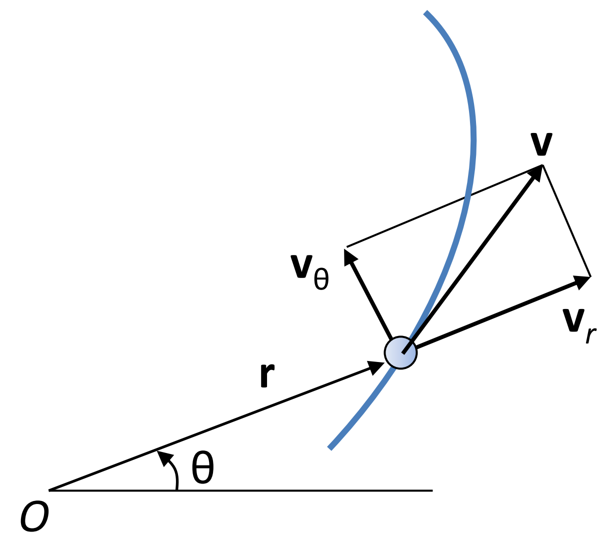
$$\dot{\mathbf{u}}_r = \dot{\theta}\mathbf{u}_\theta$$

- So we can rewrite instantaneous velocity in component form as

$$\boxed{\mathbf{v} = v_r\mathbf{u}_r + v_\theta\mathbf{u}_\theta}$$

where

$$\boxed{\begin{aligned} v_r &= \dot{r} \\ v_\theta &= r\dot{\theta} \end{aligned}}$$



Velocity

Velocity

- The speed (or magnitude of velocity) is therefore given by

$$v=\sqrt{(\dot{r})^2+(r\dot{\theta})^2}$$

Acceleration

- Differentiating the instantaneous velocity equation yields the instantaneous acceleration

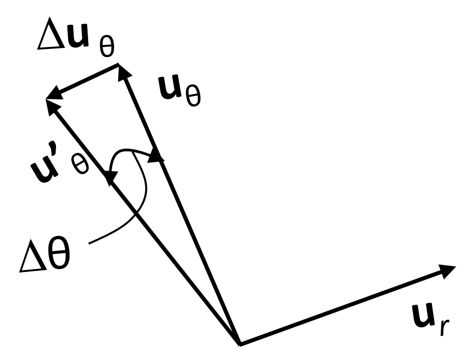
$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r}\,\mathbf{u}_r + \dot{r}\,\dot{\mathbf{u}}_r + \dot{r}\dot{\theta}\,\mathbf{u}_\theta + r\ddot{\theta}\,\mathbf{u}_\theta + r\dot{\theta}\,\dot{\mathbf{u}}_\theta$$

- We must now determine $\dot{\mathbf{u}}_\theta$
- Consider a small change $\Delta\theta$ over an interval Δt

$$\mathbf{u}_\theta^{'} = \mathbf{u}_\theta + \Delta\mathbf{u}_\theta$$

for small angles $\Delta\mathbf{u}_\theta \approx \Delta\theta$

$$\Delta\mathbf{u}_\theta = \Delta\theta\,\mathbf{u}_\theta$$



but $\Delta\mathbf{u}_\theta$ acts in the negative $\mathbf{u}_r$ direction, so we can change the unit vector for its direction

Acceleration

- $$\Delta \mathbf{u}_\theta = -\Delta\theta\, \mathbf{u}_r$$

  thus

  $$\dot{\mathbf{u}}_\theta = \lim_{\Delta t \to 0}\left(\frac{\Delta \mathbf{u}_\theta}{\Delta t}\right) = -\left(\lim_{\Delta t \to 0}\frac{\Delta\theta}{\Delta t}\right)\mathbf{u}_r$$

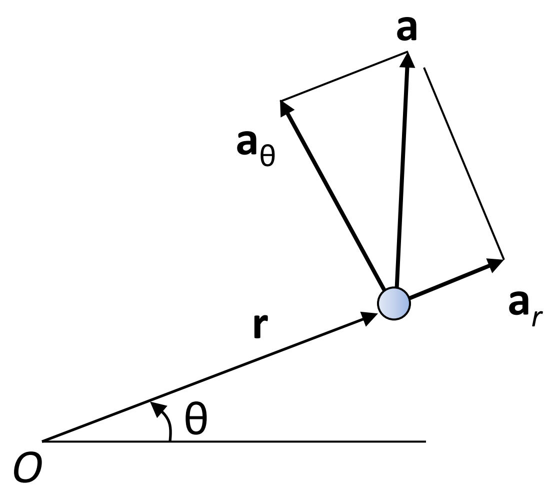
  $$\dot{\mathbf{u}}_\theta = -\dot{\theta}\,\mathbf{u}_r$$

  So we can rewrite instantaneous velocity in component form as

  $$\boxed{\mathbf{a} = a_r \mathbf{u}_r + a_\theta \mathbf{u}_\theta}$$

  where

  $$\boxed{\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{aligned}}$$



Acceleration

Acceleration

- The term $\ddot{\theta}$ is called the *angular acceleration*

$$\ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right)$$

- The *magnitude* of the acceleration

$$a = \sqrt{\left(\ddot{r} - r\dot{\theta}^2\right)^2 + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)^2}$$

- The acceleration is generally not tangential to the path.

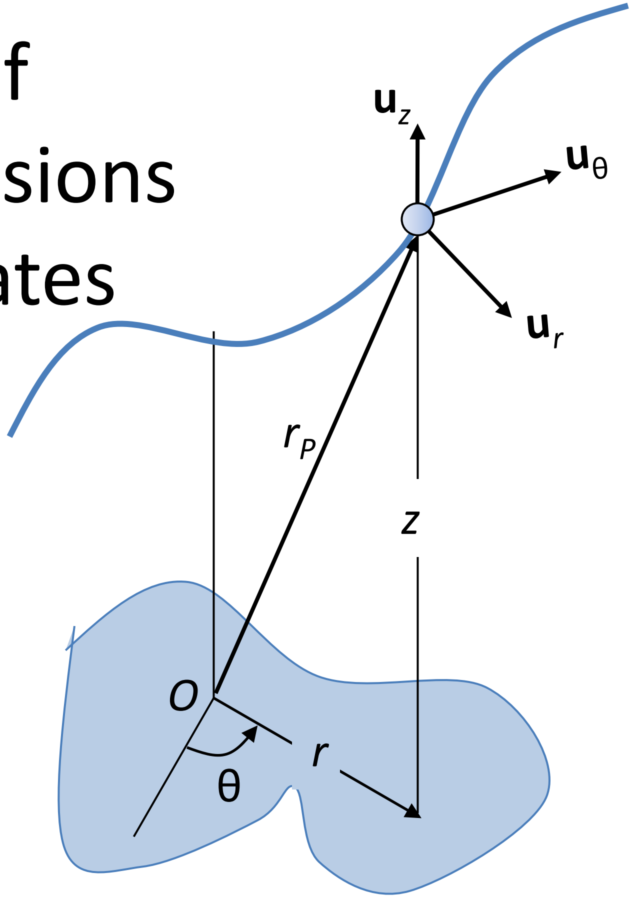
Cylindrical Coordinates

- If the particle moves along a 3-dimensional curve (also called a space curve) then we can apply cylindrical coordinates to analyze the motion



Cylindrical Coordinates

- We can extend the concept of polar coordinates to 3-dimnesions to create cylindrical coordinates *r*, θ, and *z*



$$\mathbf{r}_P = r\mathbf{u}_r + z\mathbf{u}_z$$

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{u}_z$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{u}_z$$

Questions & Comments

- Examples