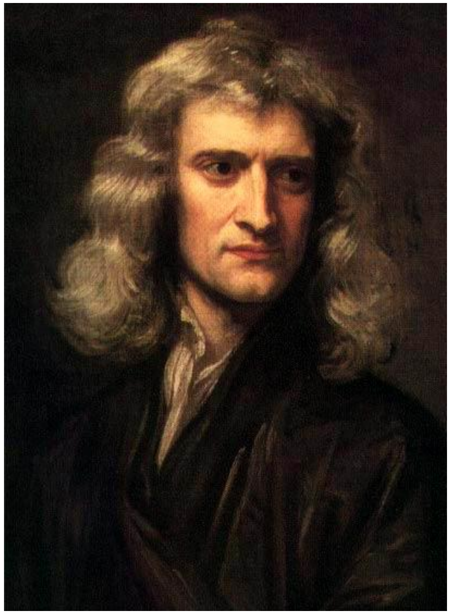
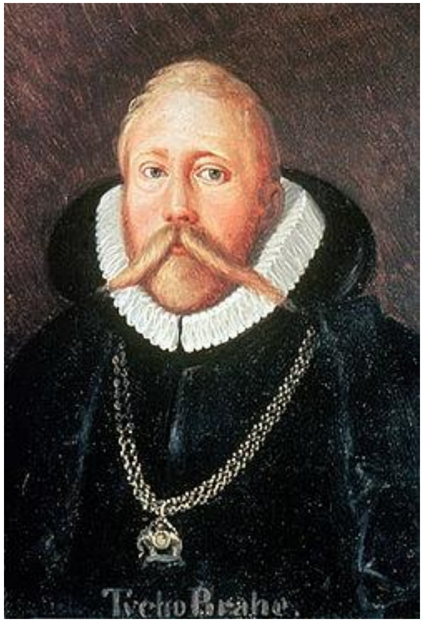


Kinetics

Central Force Motion
&
Space Mechanics

Outline

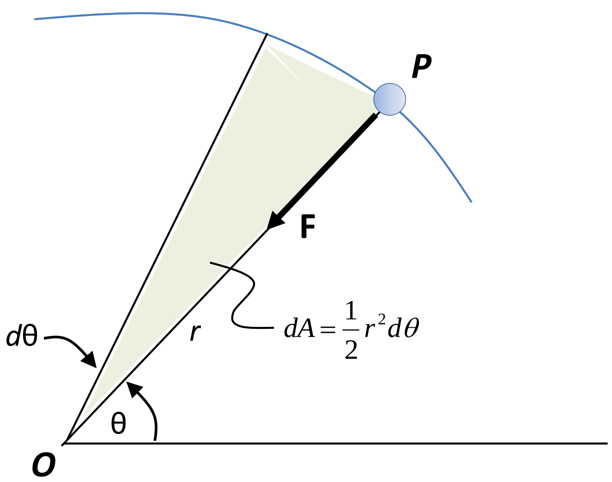
- Central Force Motion
- Orbital Mechanics
- Examples



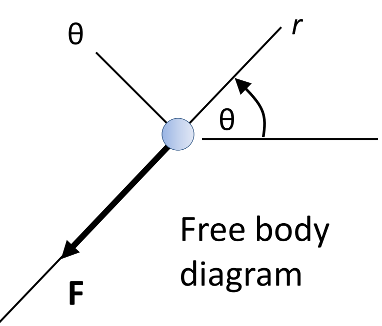
Central-Force Motion

- If a particle travels under the influence of a force that has a line of action directed towards a fixed point, then the motion is called *central-force motion*
- Examples: planetary motion, electrostatic forces, centrifuge

Central-Force Motion



- Consider a particle of mass m acted on by a force F, with center *O*
- Using the Equations of Motion for Cylindrical Coordinates, it can be shown that (derivation omitted here, student please review):



Free body diagram

$$\sum F_r = ma_r \Rightarrow -F_r = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \qquad \text{(13-11a)}$$

$$\sum F_\theta = ma_\theta \Rightarrow 0 = m\left(r\frac{d^2\theta}{dt^2} - 2\frac{dr}{dt}\frac{d\theta}{dt}\right) \qquad \text{(13-11b)}$$

Central-Force Motion

We may re-write Eqn 13-11b in the form

$$\frac{1}{r}\left[\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]=0$$

then by integration

$$r^2\frac{d\theta}{dt}=h \qquad (13\text{-}12)$$

where *h* is the constant of integration

Central-Force Motion

- Since the particle sweeps through angle *d*θ in time interval *dt*

$$dA = \tfrac{1}{2} r^2 d\theta$$

$$\frac{dA}{dt} = \tfrac{1}{2} r^2 \frac{d\theta}{dt} = \frac{h}{2}$$

- *dA/dt* is called the *areal velocity*. It remains constant for a particle in central-force motion
- This means the particle sweeps through equal areas in equal time as it travels along the path

Central-Force Motion

- Let us derive the *path of motion r* as a function of θ
- By the chain rule (Calculus 101 !)

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta}$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) = \frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\frac{d\theta}{dt} = \left[\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\right]\frac{h}{r^2}$$

- Let us denote $\frac{1}{r} = \xi$

Central-Force Motion

- So that we obtain

$$\frac{d^2r}{dt^2} = -h^2\xi^2\frac{d^2\xi}{d\theta^2}$$

- also the square of Eqn 13-12 becomes

$$\left(\frac{d\theta}{dt}\right)^2 = h^2\xi^4$$

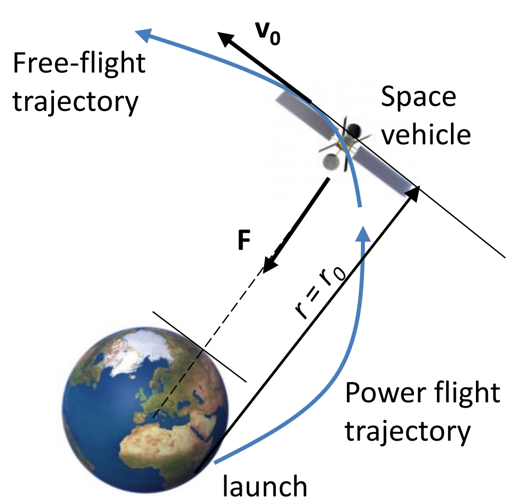
- Substituting the above in Eqn 13-11a we obtain a differential equation which can be solved to determine the *path of motion*

$$-h^2\xi^2\frac{d^2\xi}{d\theta^2} - h^2\xi^3 = -\frac{F}{m} \quad \textbf{OR} \quad \frac{d^2\xi}{d\theta^2} + \xi = \frac{F}{mh^2\xi^2}$$

(13-14)

Orbital Mechanics

- Consider a space vehicle of mass *m* launched into free-flight orbit with initial velocity $\mathbf{v}_o$.
- Assume $\mathbf{v}_o$ acts parallel to the tangent to the earth surface
- Neglect gravitational attractions of sun and moon



Orbital Mechanics

- At the instant just after release into free flight the only force acting on it is the gravitational attraction from the earth
- According to Newton's law of gravitation;

$$F = G\frac{M_e m}{r^2}$$

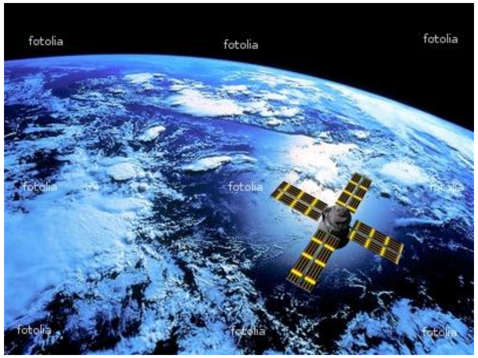
- To obtain the orbital path, substitute into Eqn 13-14

$$\frac{d^2\xi}{d\theta^2} + \xi = \frac{GM_e}{h^2}$$

Orbital Mechanics

- The above differential equation can be solved as the sum of the complementary and particular solutions (Review your Differential Equations)
- Solution is:

$$\xi = \frac{1}{r} = C\cos(\theta - \phi) + \frac{GM_e}{h^2} \qquad (13\text{-}16)$$



Orbital Mechanics

- Eqn 13-16 is the equation of a conic section [student, please review your Pre-Cal materials]
- By definition, Eccentricity

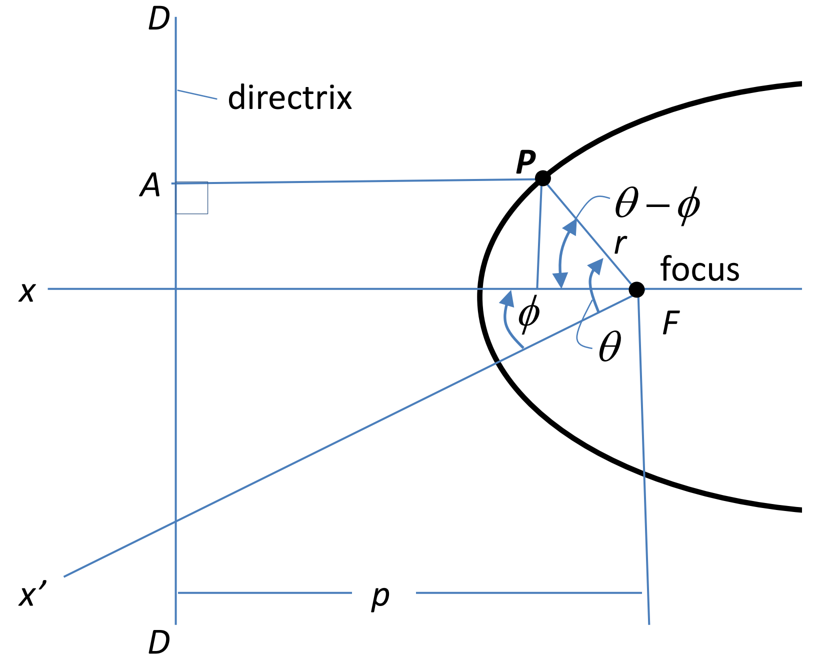
$$e = \frac{FP}{PA}$$

$$r = e(PA)$$

$$r = e[p - r\cos(\theta - \phi)]$$

$$or$$

$$\frac{1}{r} = \frac{1}{p}\cos(\theta - \phi) + \frac{1}{ep}$$



Orbital Motion

- Comparing with Eqn 13-16;

$$p = \frac{1}{C} \qquad (13\text{-}17)$$

and

$$\boxed{e = \frac{Ch^2}{GM_e}} \qquad (13\text{-}18)$$

- Provided θ is measured from the x-axis which is perpendicular to the directrix (and an axis of symmetry), then $\Phi = 0$, and Eqn 13-16 reduces to

$$\frac{1}{r} = C\cos(\theta) + \frac{GM_e}{h^2} \qquad (13\text{-}19)$$

Orbital Motion

- The constants *C* and *h* are determined from the boundary conditions at the end of the *power-flight trajectory*
- At the beginning of free-flight r = r_o, v = v_o; if θ = *Φ* = 0, then from curvilinear motion-cylindrical components

$$v_o = r_0\left(\frac{d\theta}{dt}\right)$$

$$h = r_0^2\,\frac{d\theta}{dt} \qquad \boxed{or \quad h = r_0\,v_0} \qquad (13\text{-}20)$$

Orbital Motion

- Substituting Eqn 13-20, $r = r_o$, $\theta = 0$, into Eqn 13-19

$$C = \frac{1}{r_0}\left(1 - \frac{GM_e}{r_0 v_0^2}\right) \qquad (13\text{-}21)$$

- The equation for the free-flight trajectory therefore becomes

$$\frac{1}{r} = \frac{1}{r_0}\left(1 - \frac{GM_e}{r_0 v_0^2}\right)\cos\theta + \frac{GM_e}{r_0 v_0^2} \qquad (13\text{-}22)$$

Orbital Motion

- The type of path traveled by the space vehicle depends on the value of the eccentricity

$$\boxed{\begin{array}{lll} e=0 & \Rightarrow & \text{circle} \\ e=1 & \Rightarrow & \text{parabola} \\ e<1 & \Rightarrow & \text{ellipse} \\ e>1 & \Rightarrow & \text{hyperbola} \end{array}} \qquad (13\text{-}23)$$

- [Students, plug in these values to the appropriate equations and verify these conclusions]

Orbital Motion

- Parabolic path: The spacecraft is on the borderline of never returning to its starting point.
- The initial velocity required for a parabolic path is called the escape velocity
- Plugging e = 1, Eqns 13-21 and 13-22 into Eqn 1318;

$$v_e = \sqrt{\frac{2GM_e}{r_0}} \qquad (13\text{-}24)$$

Orbital Motion

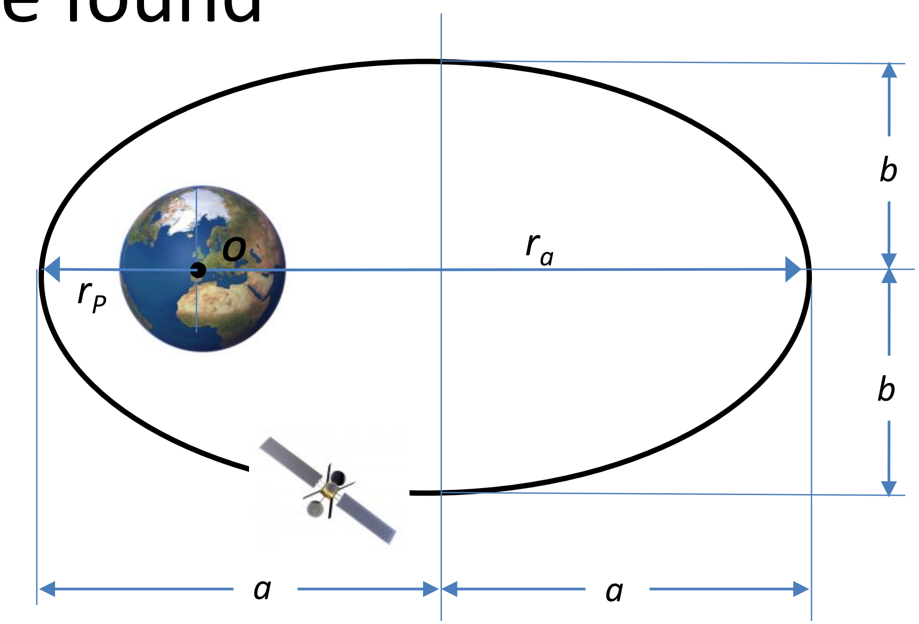
- Similarly, for Circular Motion

$$\boxed{v_c = \sqrt{\frac{GM_e}{r_0}}} \qquad (13\text{-}25)$$

- Note that $\mathbf{v}_o \geq v_e$ will result in vehicle escaping earth's gravitational pull
- On the other hand if $\mathbf{v}_o < v_c$ the vehicle will fail to reach orbit, renter earth atmosphere, and crash or burn up in the heat of reentry

Elliptical Orbit

- All planets and most artificial satellites orbit in an elliptical path.
- For the space craft the minimum distance to the center of the earth (with earth at a focus of the ellipse), r_p can be found by plugging θ = 0 into Eqn 13.22
- Using θ = 180° we get the max distance r_a



Elliptical Orbit

$$\boxed{r_p = r_0} \qquad (13\text{-}26)$$

- r_p is called the perigee (generally periapsis)

$$\boxed{r_a = \frac{r_0}{(2GM_e / r_0 v_0^2) - 1}} \qquad (13\text{-}27)$$

- r_a is called the apogee (generally apoapsis)
- Half the length of the major axis

$$\boxed{a = \frac{r_p + r_a}{2}} \qquad (13\text{-}28)$$

Elliptical Orbit

- It can also be shown that

$$\boxed{b = \sqrt{r_p r_a}} \qquad (13\text{-}29)$$

(Students, verify on your own)

- By integration, the area of the ellipse is

$$\boxed{A = \pi ab = \frac{\pi}{2}(r_p + r_a)\sqrt{r_p r_a}} \qquad (13\text{-}30)$$

Elliptical Motion

- The areal velocity was defined in Eqn 13-13 as

$$\frac{dA}{dt} = \frac{h}{2} \quad \textbf{or} \quad \int \frac{dA}{dt} = \int \frac{h}{2}$$

$$\Rightarrow A = \frac{hT}{2}$$

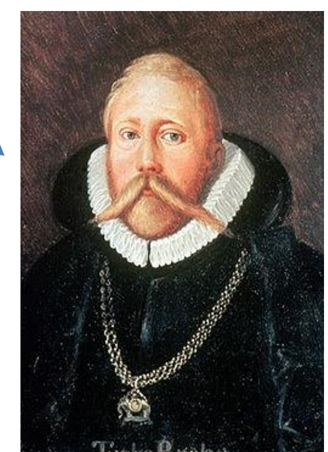
where *T* is the time to make one orbital revolution (aka orbital period).

- From 13-30:

$$\boxed{T = \frac{\pi}{h}(r_p + r_a)\sqrt{r_p r_a}} \qquad (13\text{-}31)$$

Laws of Planetary Motion

- The theory developed in this chapter was first presented by Johannes Kepler in 1621, 6 clear decades before Newton's *Principia*
- Kepler developed the laws of planetary motion over 20 years by studying planetary data collected by his mentor Tycho Brahe



Laws of Planetary Motion

- Planets travel in elliptical orbits with the sun at a focus of the ellipse (Eqn 13-22)
- Planets travel in an orbit such that they sweep equal areas in equal time intervals (Eqn 13-13)
- The square of the period of any planet is directly proportional to the cube of the major axis of its orbit (Eqns 13-31, 13-19, 13-28, 13-29)

Questions & Comments