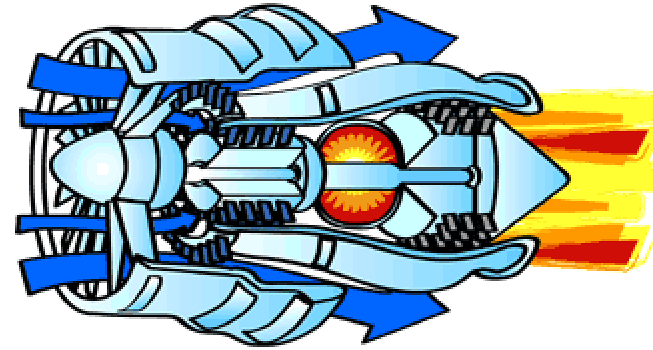


Kinetics of a Particle

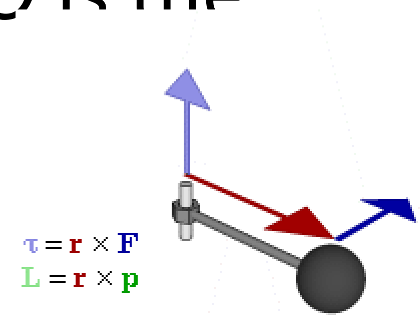
Impulse & Momentum II

Outline

- Angular Momentum
- Steady Flow of a Fluid Stream
- Jet Propulsion

Angular Momentum

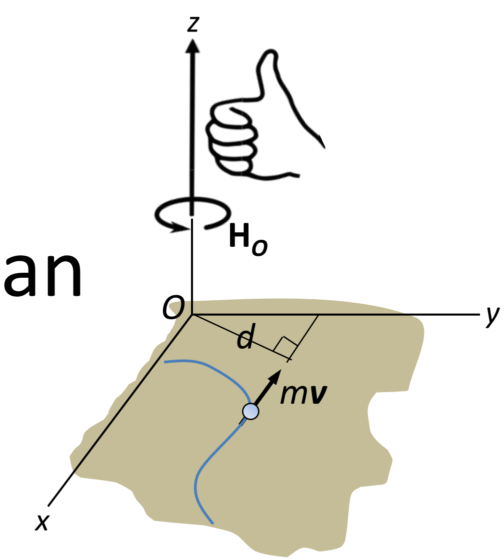
- *Angular momentum* about a point *O* is the moment (**H**$_O$)of the particles linear momentum about *O*.



- Magnitude of Angular Momentum:

$$(H_O)_z = (d)(mv) \quad \text{(15-12)}$$

- The sense of the rotation of *m**v*** can be obtained using the right hand rule



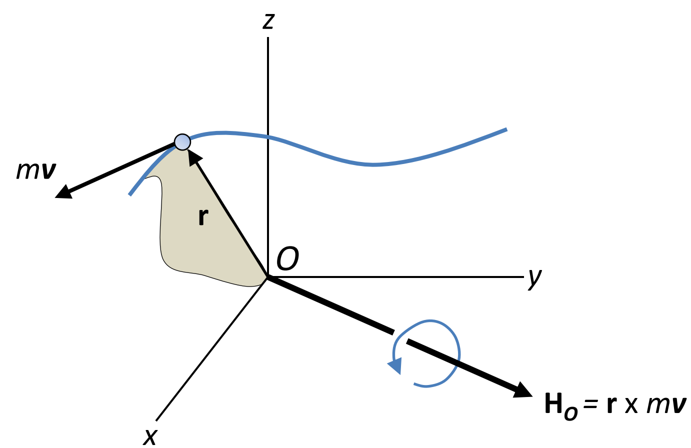
Angular Momentum - Vector

- If a particle moves along a space curve then the angular momentum (vector) can be determined using the vector cross product about *O*.

$$\boxed{\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}} \qquad (15\text{-}13)$$

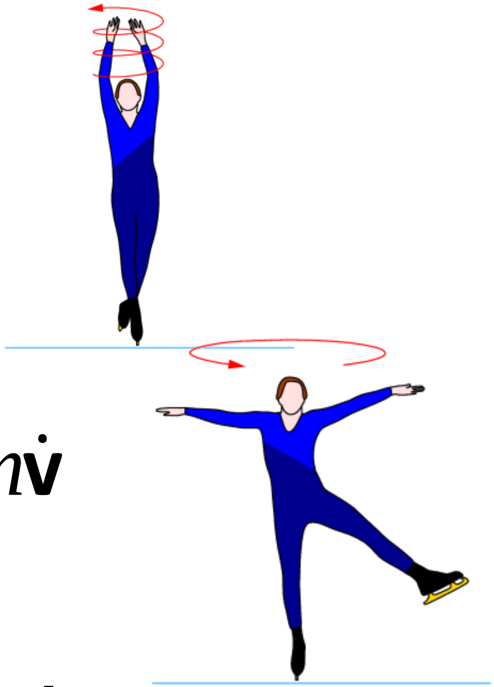
- The cross product is evaluated as the determinant of

$$\boxed{\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix}} \qquad (15\text{-}13)$$



Moment of a Force Versus Angular Momentum

- From Newton's 2nd law, $$\sum \mathbf{F} = m\dot{\mathbf{v}}$$



- Evaluating the moments

$$\sum \mathbf{M}_O = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times m\dot{\mathbf{v}}$$

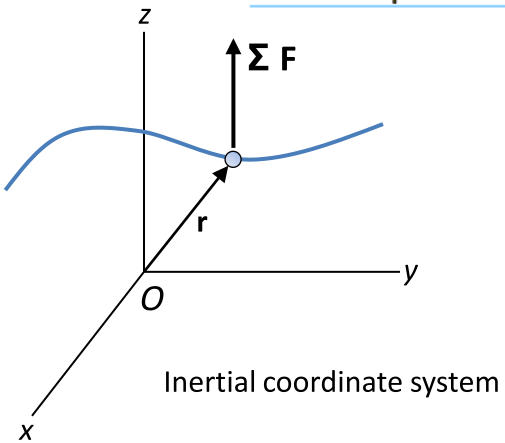
- Also

$$\dot{\mathbf{H}}_O = \frac{d}{dt}\left(\mathbf{r} \times m\mathbf{v}\right) = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}}$$

- Now

$$\dot{\mathbf{r}} \times m\mathbf{v} = \dot{\mathbf{r}} \times m\dot{\mathbf{r}} = m\left(\dot{\mathbf{r}} \times \dot{\mathbf{r}}\right) = 0$$

- So $$\boxed{\sum \mathbf{M}_O = \dot{\mathbf{H}}}$$ (15-15)



Moment of a Force Versus Angular Momentum

- So the resultant moment about point *O* of all forces acting the particle equals the rate of change of the particle's angular momentum about point *O*.
- Interestingly, this is the same result of Eqn 15-1 (definition of linear momentum)
- Eqns 15-1 and 15-15, are alternate ways of expressing Newton's 2nd law. We shall see how this enables solve problems later in the topic rigid bodies

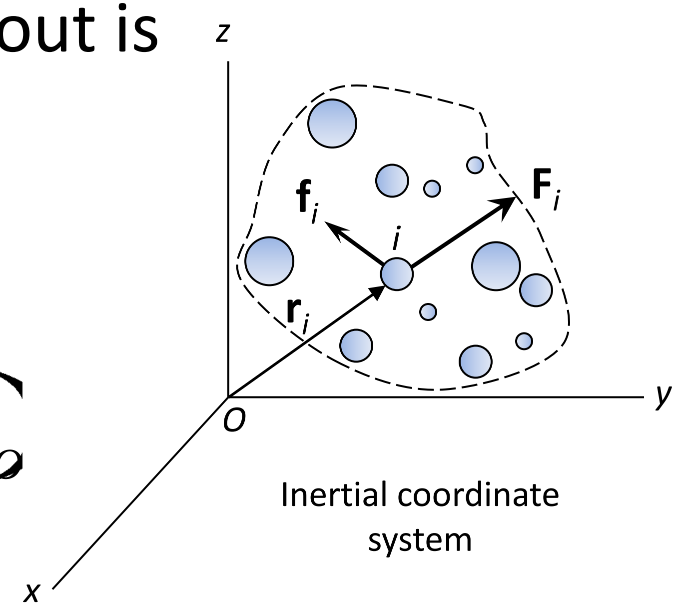
System of Particles

- We may extend the previous analysis to a system of particles.
- Consider the i^{th} particle subject to a resultant external force $\mathbf{F}_i$, and a resultant internal force $\mathbf{f}_i$
- From Eqn 15-15, the rate of change of angular momentum of the i^{th} particle about is

$$(\mathbf{r}_i \times \mathbf{F}_i) + (\mathbf{r}_i \times \mathbf{f}_i) = (\dot{\mathbf{H}}_i)_O$$

- Summing for all particles in this system

$$\sum (\mathbf{r}_i \times \mathbf{F}_i) + \sum (\mathbf{r}_i \times \mathbf{f}_i) = \sum (\dot{\mathbf{H}}_i)_O$$

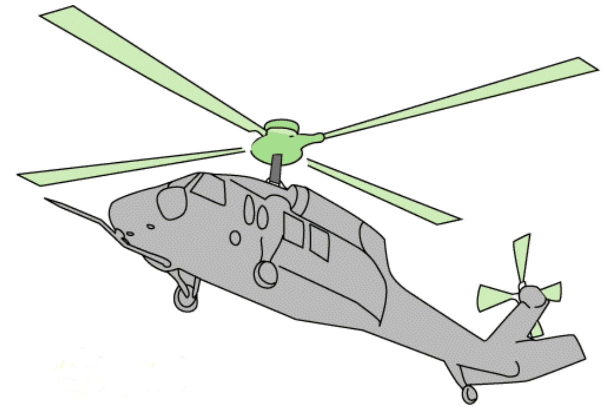


Inertial coordinate system

System of Particles

- Since internal forces will essentially cancel each other out,

$$\sum (\mathbf{r}_i \times \mathbf{F}_i) = \sum (\dot{\mathbf{H}}_i)_O$$



- So for the system

$$\sum \mathbf{M}_O = \dot{\mathbf{H}}_O \qquad (15\text{-}17)$$

- In other words, the sum of moments about *O* of all the external forces acting on the system of particles equals the rate of change of the total angular momentum of the system about *O*.

Principle of Angular Impulse and Momentum

- Let us rewrite Eqn 15-15 in the form $\sum \mathbf{M}_O dt = d\mathbf{H}_O$ and integrate from time t_1 to t_2 which have corresponding angular momenta $(\mathbf{H}_o)_1$ and $(\mathbf{H}_o)_2$

$$\sum \int_{t_1}^{t_2} \mathbf{M}_O dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1$$

$$\boxed{(\mathbf{H}_O)_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_O dt = (\mathbf{H}_O)_2} \qquad (15\text{-}18)$$

- This is the *principle of angular impulse and momentum*.
- The second term on the LHS is the *angular impulse*.

Angular Impulse

- Angular impulse

$$\sum \int_{t_1}^{t_2} \mathbf{M}_O \, dt = \int_{t_1}^{t_2} (\mathbf{r} \times \mathbf{F}) \, dt \qquad (15\text{-}19)$$

- By extension, we may apply the principle of angular momentum to a system of particles

$$\sum (\mathbf{H}_O)_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_O \, dt = \sum (\mathbf{H}_O)_2 \qquad (15\text{-}20)$$

- where

$$\sum (\mathbf{H}_O) = \sum (\mathbf{r}_i \times m\mathbf{v}_i)$$

Angular Momentum

- We may rewrite a vector formulation of Eqn 15-3 and Eqn 15-18 using the impulse and momentum principles

$$m\mathbf{v}_1 + \sum \int_{t_1}^{t_2} \mathbf{F}\, dt = m\mathbf{v}_2$$

$$(\mathbf{H}_O)_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 \qquad (15\text{-}21)$$

Angular Momentum – Scalar Formulation

- Eqn 15-21 may be expressed in *x*, *y*, and *z* components. For the *x* – *y* plane we have the following

$$m(v_x)_1 + \sum \int_{t_1}^{t_2} F_x\,dt = m(v_x)_2$$

$$m(v_y)_1 + \sum \int_{t_1}^{t_2} F_y\,dt = m(v_y)_2 \qquad (15\text{-}21)$$

$$(H_O)_1 + \sum \int_{t_1}^{t_2} M_O\,dt = (H_O)_2$$

Conservation of Angular Momentum

- When the angular impulses on a particle are zero from time t_1 to time t_2, from Eqn 15-18,

$$\boxed{(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2} \qquad (15\text{-}23)$$

- For a system of particles

$$\boxed{\sum (\mathbf{H}_O)_1 = \sum (\mathbf{H}_O)_2} \qquad (15\text{-}24)$$

- The above equations represent the principle of conservation of angular momentum.
- If no external impulse acts on a particle, linear and angular momentum will be conserved. In some cases angular momentum will be conserved by linear momentum may not. E.g central force motion

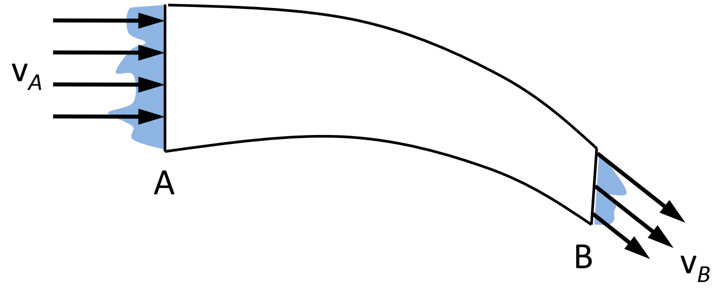
Questions

Steady Flow of Fluid Stream

- We shall now apply impulse and momentum principles to fluid flowing through a pipe, under the following conditions:

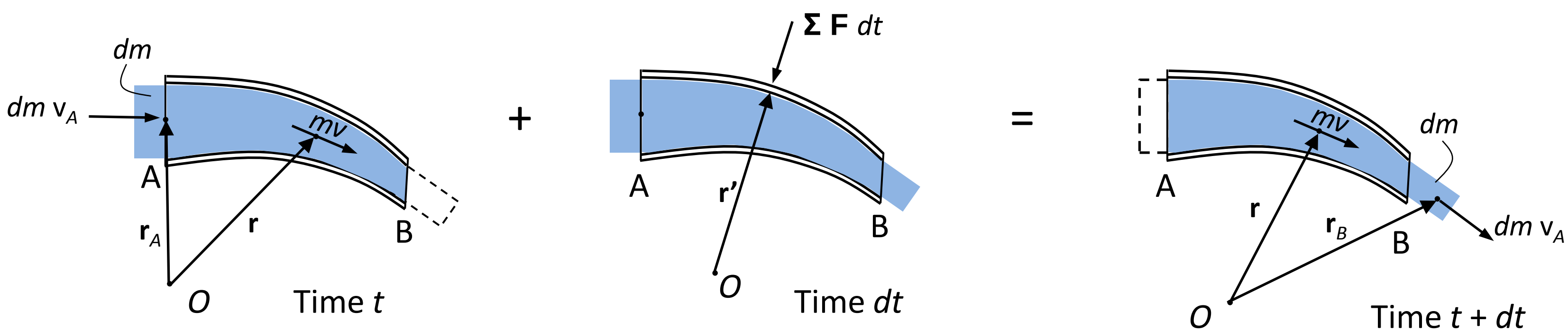
1. The mass of fluid flowing into and out of the *control volume* are equal.
2. The size and shape of the control volume coincides with the internal boundaries and openings of the pipe.

- If the above are met the fluid has *steady flow*.



Steady Flow of a Fluid Stream

- Consider a small amount of fluid of mass dm about to enter the control volume at A with v_A at time t
- Due to steady flow the same amount of fluid that entered over dt will leave at B, with v_A leaving at B
- The momenta of the fluid entering and leaving will be $dm\ v_A$ *and* $dm\ v_B$ respectively
- Momentum of the fluid inside control volume remains constant over dt



Steady Flow of a Fluid

- The resultant external force on the pipe stream (from the reactive force from the pipe wall) produces an impulse $\Sigma\,\mathbf{F}\,dt$
- Applying the principle of linear impulse and momentum

$$dm\,\mathbf{v}_A + m\mathbf{v} + \sum \mathbf{F}\,dt = dm\,\mathbf{v}_B + m\mathbf{v}$$

- $\mathbf{r}$, $\mathbf{r}_A$, and $\mathbf{r}_B$ denote the position vectors of the centers of the control volume, and openings at A and B respectively. Therefore by principle of angular impulse and momentum

$$\mathbf{r}_A \times dm\,\mathbf{v}_A + \mathbf{r} \times m\mathbf{v} + \mathbf{r}' \times \sum \mathbf{F}\,dt = \mathbf{r}_B \times dm\,\mathbf{v}_B + \mathbf{r} \times m\mathbf{v}$$

Steady Flow of a Fluid

- Diving both sides by *dt*

$$\boxed{\sum \mathbf{F} = \frac{dm}{dt}(\mathbf{v}_B - \mathbf{v}_A)} \qquad \text{(15-25)}$$

$$\boxed{\sum \mathbf{M}_O = \frac{dm}{dt}(\mathbf{r}_B \times \mathbf{v}_B - \mathbf{r}_A \times \mathbf{v}_A)} \qquad \text{(15-26)}$$

- The term *dm/dt* is called the *mass flow*

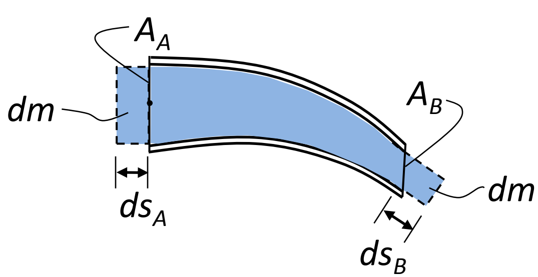
Steady Flow of a Fluid

- A_A denotes the cross section area of the control volume at A. Let ρ_A denote the density of the fluid at A.
- For an incompressible fluid, the continuity of mass requires that $dV = dm/\rho$ or $dm = \rho dV$

$$dm = \rho_A dV_A = \rho_A ds_A A_A$$

- And likewise for any other point of interest. Therefore, in general

$$\boxed{\frac{dm}{dt} = \rho v A = \rho Q} \qquad (15\text{-}27)$$

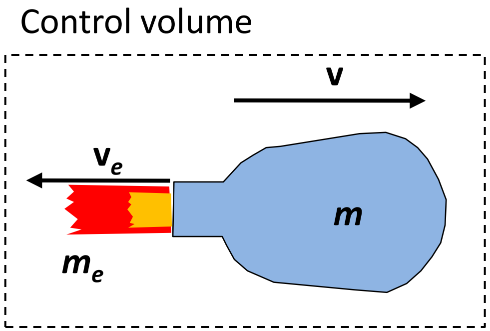


- Q is called the discharge, or volumetric flow rate

Questions

Propulsion With Variable Mass

- Consider a rocket of mass m at an instant in time, moving with velocity v.
- At the same instant an amount of mass m_e is expelled with velocity v_e.
- The control volume in this case is that of both m and m_e.
- During an interval dt, velocity increases from v to v + dv, due to loss of mass dm_e that is ejected. However v_e remains constant from the view of a stationary observer.



Propulsion with Variable Mass

- Applying the Principle of impulse and momentum to the control volume between time *t* and time *t* + *dt*,

$$mv - m_e v_e + \sum F_{cv} dt = (m - dm_e)(v + dv) - (m_e + dm_e) v_e$$

- or

$$\sum F_{cv} dt = -v dm_e + m dv - dm_e dv - v_e dm_e$$

- Now $dm_e dv \approx 0$ and dividing by *dt*

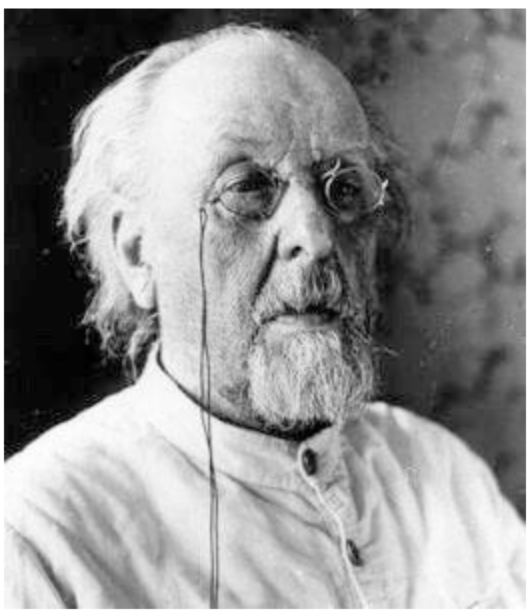
$$\sum F_{cv} = m \frac{dv}{dt} - (v + v_e) \frac{dm_e}{dt}$$

- The velocity of the rocket as observed from a rider “riding’ on the exhaust stream is $v_{D/e} = v + v_e$, therefore

$$\boxed{\sum F_{cv} = m \frac{dv}{dt} - v_{D/e} \frac{dm_e}{dt}} \quad (15\text{-}28)$$

Propulsion with Variable Mass

- Note that the second term on the RHS is the rate of mass ejection
- Eqn 15-28 is the famous Tsiolkovsky rocket equation or ideal rocket equation, after Konstantin Tsiolkovsky (Russia) who published it in 1903.



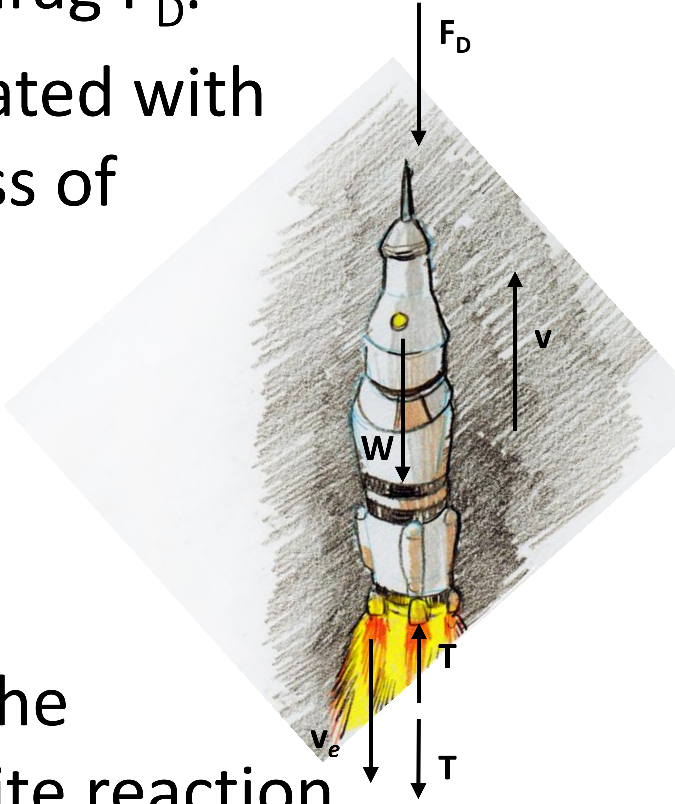
Hermann Oberth

Robert Goddard

Founding "Fathers" of Rocketry and Astronautics

Propulsion with Variable Mass

- Consider a rocket of weight W moving upwards with velocity v, subject to atmospheric drag F_D.
- If the control volume is that associated with the mass of the rocket and the mass of ejected gas m_e
- From Eqn 15-28



$$-F_D - W = \frac{W}{g}\frac{dv}{dt} - v_{D/e}\frac{dm_e}{dt}$$

- The last term on the RHS is called the *thrust* (T), it is the equal but opposite reaction of the force exerted by ejecta on the surroundings

Propulsion with Variable Mass

- Also we know that *dv/dt* = a, so

$$\boxed{T - F_D - W = \frac{W}{g} a}$$

- Or

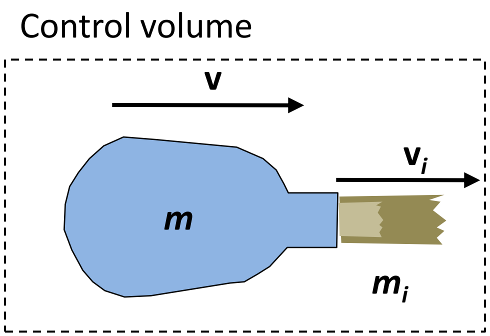
$$\boxed{\sum F = ma}$$

- So yes, Newton's 2nd law applies and is consistent with the ideal rocket equation.

Control Volume that Gains Mass

- In some applications the control volume will rather gain mass
- Example: scoop or blade of mechanical excavator
- The analysis is identical save the increase in mass rather than the decrease as previously learned

Control Volume that Gains Mass



- If v_i is the velocity of injected mass and $v > v_i$.
- Using the same procedure

$$mv + m_i v_i + \sum F_{cv} dt = (m + dm_i)(v + dv) + (m_i - dm_i) v_i$$

- Resulting in

$$\sum F_{cv} = m \frac{dv}{dt} + (v - v_i) \frac{dm_i}{dt}$$

$$\boxed{\sum F_{cv} = m \frac{dv}{dt} + v_{D/i} \frac{dm_i}{dt}} \qquad (15\text{-}29)$$

Control Volume that Gains Mass

- dm_i/dt is the rate of mass injected into the device.
- The last term on the RHS represents the magnitude of force (R) which the injected mass exerts on the device, so

$$\boxed{\sum F_{cv} - R = ma}$$

Questions